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CONTENTS

April, 1960

Papers

	Page
The Impact of Computers on Engineering Education by Gordon P. Fisher	1
Distribution of Extreme Winds in the United States by H. C. S. Thom	11
Effect of Floor Systems on Pony Truss Bridges by Robert M. Barnoff and William G. Mooney	25
Dynamic Effects of Earthquakes by Ray W. Clough	49
Discontinuity Stresses in Beams on Elastic Foundations by Joseph Penzien	67
Discussion	95

(Over)

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Page

Design of Prestressed Concrete Beams by Computer
by Joseph J. Bonasia. 129

Digest of the Guide to Design Criteria for Metal Compression Members
by Bruce G. Johnston 157

Preliminary Analysis of Continuous Gable Frames
by James W. Gillespie and Jan J. Tuma 189

Journal of the
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THE IMPACT OF COMPUTERS ON ENGINEERING EDUCATION[a]

By Gordon P. Fisher,[1] F. ASCE

SYNOPSIS

The modern electronic computer is having a significant effect on the engineering profession. This effect, naturally, is also felt in the engineering schools. In this paper there is examined the influence of the computer on (a) curriculum, (b) research, and (c) relations with engineering practice.

INTRODUCTION

The interest and attention of engineers in recent years have been caught by the startling prospect of a machine which not only will perform designs automatically and permit more accurate analysis, but do these things with great speed and with nearly complete relief from tedious and repetitive calculations. It is little wonder then that this machine, the modern electronic computer, is having a great influence on the engineering profession. This paper covers but a single and particular aspect of this influence—the impact of the electronic computer on engineering education. While the influence of computers on engineering practice has been quite significant and substantial, it is

Note.—Discussion open until September 1, 1960. To extend the closing date one month, a written request must be filed with the Executive Secretary, ASCE. This paper is part of the copyrighted Journal of the Structural Division, Proceedings of the American Society of Civil Engineers, Vol. 86, No. ST 4, April, 1960.

[a] Presented at the November 1958 ASCE Conference on Electronic Computation in Kansas City, Mo.

[1] Assoc. Prof. of Civ. Engrg., Cornell Univ., Ithaca, N. Y.

only now that it is beginning to be felt broadly in educational circles. Thus, this appraisal is not so much a matter of recording known progress to date as it is one of speculation on the future. It is an attempt, limited by one's flair for philosophy and prophecy, to define some of the more apparent emerging problems and, where possible, to consider how to meet them with at least a skeletal policy of procedure. The University has played a uniquely important role in the invention and development of the electronic computer. Today the University finds itself in the position of being forced to respond to the profound effects of this machine, and it is rightly concerned that its response be geared to the demands of the future.

The University's own needs in research and its obligations to supply talent for this new skill and industry it has spawned require an adjustment of traditional operations. Both of these motives are reflected in practically every University function. The various facets of this general challenge will be grouped arbitrarily into five areas of concern:

1. Curriculum;
2. Research;
3. Relations with engineering practice;
4. Effects on and from other areas within the University; and
5. The computer center.

It will be readily apparent that some of the things to be discussed will fall equally well under several of these subdivisions. The present discussion is restricted to digital computers, or properly speaking, electronic automatic stored-program digital computers, which is to be understood whenever the term "computer" is used. While most of the examples to be cited are drawn from structural engineering, it is hoped that the viewpoints so limited will apply or at least afford extrapolation to engineering broadly.

CURRICULUM

Because of the many philosophical dilemmas which arise, curricular problems will require the largest share of attention. The overriding question concerning curricular policy would seem to be a decision as to whether major emphasis shall be placed on educating users of computing equipment or designers of computing equipment. Since the number of users is certain to surpass enormously the number of designers, the direction is clearly in favor of the former. What is required is more fundamental education for imaginative use of computers and less attention to detailed design of actual computer hardware. The computer is essentially a tool, and the teaching of the use of tools most remain subordinated to the teaching of principles. This fact alone sets a limit on how much can and should be done in teaching computer techniques.

It is not the duty of the University to produce finished specialists in computer design, although a few institutions with the specialized facilities and faculty may find this to be a stimulating side-interest. Rather, those undergraduate students interested in design and manufacture of computers should be provided with a general fundamental program in electronics, solid-state physics, high-speed circuitry and switching theory, mathematical logic and number theory, and other subjects necessary to technical competence. But such a program should be vested with the idea that proficiency in design can

be acquired by those who wish it by later experience in the computer industry. In addition, the student should be given two things:

(1) the means for a broad appreciation of those new and promising scientific devices or principles which may form the basis for future developments in design of computers, and
(2) an opportunity to become acquainted with users and users' problems.

There is no reason to add any new technical course to the curriculum. A reorientation of subject matter within the present framework of formal technical courses should be sufficient, and in fact, much of what is required is already available. What is suggested here is a judicious pruning of course material to make room for new concepts and for firming up of fundamental ideas underlying computer design. Furthermore, whatever is taught that is especially useful to computer designers should be, at the same time, sufficiently general for wide applicability to other allied interests.

In this sense, the needs of the computer user are not really contrary to those of the designer. To be sure, the potential designer must have much more specific technical information than the user, but for both, the general principle holds true: a healthy breadth of scientific understanding fitting to the particular field of study, with computer aspects incorporated as sidelights within the regular academic program. The major change in the educational program for the user should be a shift in the attitudes of the teaching staff. As many professors as possible should inform themselves of the potentialities of computers, and this knowledge cannot help but have a natural and beneficial effect on the material taught and the manner in which it is presented. This viewpoint has been ably stated by A. S. Householder,[2] as follows:

> "---the important and lasting effect ---, the effect which it seems to me we should emphasize and encourage, is not to be measured in terms of the numbers of courses dealing directly with computers nor with the number of graduates who have had such courses. It is rather to be measured in terms of organization and accents exhibited in the standard courses. More important than the number of your graduates who can draw a flow chart is the number of your instructors who have had direct computational experience, even if only a little."

It follows readily that a formal course devoted soley to programming, coding, and other operational techniques is essentially vocational or job-training in nature, and is not academic nor particularly professional. It is therefore not desirable, although auxiliary short courses, seminars, and colloquies outside the regular program can be beneficial. On the other hand, in spite of an already crowded curriculum, serious consideration should be given to the addition on the undergraduate level of a thorough and demanding course devoted to numerical analysis, and this course should include, for maximum effectiveness, actual machine computation. Beyond this, the natural inquisitiveness of good students stimulated by the encouragement and example of a faculty knowledgeable in computer use and by the opportunity for close personal contact with computers, should be all that is necessary.

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2 "The Effect of Computers on the Training of Applied Mathematicians and Scientists," by A. S. Householder, Proc. of the First Conference on Training Personnel for the Computing Machine Field, Wayne Univ., Detroit, by A. W. Jacobson, Wayne University Press, 1955.

The unifying element linking the computer applications of many diverse fields of engineering is the universality of mathematics. Many educators foresee an unprecedented mathematization of engineering in coming years and are beginning to urge that undergraduate training in mathematics go well beyond the usual terminus of differential equations. Such an improvement in general mathematical scope and ability, a trend which is developing quite independently of the demands of computer usage, will be of great benefit. However, much can be done with those courses already in the curriculum. It is inconceivable, for example, that an adequate course in differential equations can be limited to solution of a few types of equations in closed form and ignore a systematic examination of numerical methods of solution for other types, those types that eventually may prove to be of the greatest interest and applicability for engineers. Certainly any increased mathematical instruction should give considerably more attention than at present is accorded to matrix algebra beyond the simple aspects of determinants taught in Freshman mathematics courses. Likewise, some attention must be given to iteration or relaxation methods, not only applications but especially the underlying mathematical principles.

The advantages of matrix analysis and iteration in the solution of simultaneous equations, especially with computers, make it a rather natural response to give preference or at least equal emphasis in technical courses to those theories or methods which lead to simultaneous equations. This response is particularly natural when the effort involved in writing the equations is small. In fact, this begins to be the case. For example, it has been a fashion until recently in teaching structural theory to de-emphasize, almost to exclusion, the method of slope deflection in favor of moment distribution for rigid frame analysis. This relative dominance was assigned for the very practical reason that moment distribution usually allows great time-savings in manual computation by avoiding the burden of solving simultaneous equations. But now, the advent of machine computation has seen the slope-deflection method dusted off and ranked in importance on a par with moment-distribution. And there is an extra dividend: a thorough treatment of slope deflection reveals its essential kinship with the classical analysis method known as consistent deformation, the practical difference being largely whether one selects the loads or the deformations as the unknowns. A shift in accents similar to that described, often with increased depth of understanding, seems possible in most areas of engineering.

Even the principles underlying some of our classical theories will require some re-examination in the light of secondary problems that arise in computer solutions. A case in point is the recent increase of concern with ill-conditioned simultaneous equations, whose solution may go wildly erratic or divergent. At best, the solution may be so much in error as to render the solution either worthless for practical use or hardly worth the effort of programming it for the computer. Special attention has been given to detecting such situations and predicting in advance of programming the magnitude of the error to be expected.[3] But beyond these purely mathematical consequences of ill-condition, it is natural to seek for deeper reasons within the theoretical physical approach itself that may have given rise to such equations in the first place. Again falling back on structural theory for example, a previous observation may be recalled that in framework analysis either the forces or the de-

[3] "Ill-Conditioned Equations and Matrices," by R. B. McCalley, Jr., presented at ASCE Conference on Electronic Computation, Kansas City, Mo., November, 1958.

formations may be selected as the unknowns. It has been suggested by Southwell[4] and others that Castigliano's Method of Least Work may often result in ill-conditioned equations and yield answers whose unreliability is not readily deduced.

The least-work method is a strain energy method which employs as unknowns the forces in arbitrarily selected members and employs as coefficients the corresponding displacements. One might test for ill-condition by making small changes in the displacements and observing the resulting changes in the forces, but deciding whether the changes are acceptable from a design viewpoint is always subject to considerable uncertainty. On the other hand, if definite external forces are selected as the constant coefficients and the displacements as the wanted quantities, as in applying Rayleigh's Principle of Minimum Potential Energy, it is extremely unlikely that ill-conditioned equations will result for a competently designed framework. For any unsuitable framework, ill-condition would be indicated by large changes in the displacements as a result of slight changes in the external loads. But in this case, it is much easier to judge that the displacements are unacceptable for design. Thus, the latter method seems preferable.

The example cited is quite typical in that the re-examination of fundamental aspects as described really does not involve the computer at all. Rather, the problem has become an object of general concern largely through a wider experience with complex systems, systems of a type seldom attacked before computers were available. This example is but one illustration of the sort of revision of basic ideas that may be prompted by use of computers and of an influence which certainly will be reflected in teaching the related subject matter.

The examples developed here are intended to be indicative rather than exhaustive, but should serve to affirm the significant influence of the digital computer on engineering curricula. The key proposition expressed is that changes in curricula necessary to meet the needs of both designers and users of computers should be largely shifts of accent rather than addition of courses directly concerned with computers. This reorientation will depend for maximum effectiveness and vitality on two things:

1. Emphasis on increased mathematical ability, particularly in the direction of numerical analysis, and
2. Wider personal participation of teaching staff in machine computation.

It is not the intention here to discourage the addition of fundamental courses where necessary. That this need is rapidly becoming more urgent has been apparent for some time, and the digital computer is yet another pressure in the same direction. From this viewpoint, reorientation of subject matter within the present curricular framework is only a minimum answer rather than the optimum. We ask for increased mathematical ability, we ask for increased sophistication in basic sciences, and yet we wish to retain equally desirable technical and humanities courses. All this in four years! The net result is a dilemma for the engineering curriculum planner. It is high time that engineering educators give serious consideration to extending the period of undergraduate education. Engineering is the only profession attempting to educate in four years. The others—law, medicine, the ministry—

[4] "Current Trends in Structural Research," by R. V. Southwell, Research, Engrg. Structures Supplement, Colston Papers, Vol. II London, 1949, p. 1.

require six to eight, exclusive of graduate study and internship. At Cornell University, where there is a five-year undergraduate program, some ten years' happy experience has revealed that its adoption was a wise and far-sighted decision.

RESEARCH

The second aspect of computer influence to be discussed is university research. The teaching program of a university has natural ties to its research efforts and indeed, depends upon the research activities of professors for much of its enrichment. The computer is proving to be a versatile tool which is expanding tremendously the potential scope of research. Many examples come to mind. Complex and repetitive calculations formerly prohibitive in terms of computation time are now handled with ease and rapidity. This new ability has helped to bolster the quality and scope of graduate work, particularly research theses which often suffer in the student's attempt to meet a June deadline. There is no longer any reason for graduate students to eliminate interesting problems as thesis topics solely because of the prospect of too much computation. Large amounts of data may be quickly reduced to averages and meaningful answers. Fewer assumptions and simplifications need be made; analysis may be more complex and design more refined with the same or less work than ever before. Difficult theoretical problems may be approximated by simple mathematical models and so lead step by step to a deep and detailed understanding. Experiments may be enlarged in scope and based on more complete and comprehensive data.

Indeed, the computer is especially beneficial in experimentation. It may be used to simulate processes or equipment, predicting their performance without ever building the prototype. Many devices, such as jet engines and nuclear reactors, may be designed gradually through several preliminary phases without costly prototype experiments. Machinery and structures may be optimized to find the best combination of possibly dozens of design variables in terms of final performance and cost. Optimizing by digital computer has been a boon in highly empirical situations where trial and error solution is mandatory, where previously the limited number of trials economical in manual computation gave low confidence levels in design.

One might go on citing example upon example, but these few are sufficient to show that the possibilities in research are limited only by the versatility of the human mind. It is clear that a high premium is placed on intuition and physical understanding. The computer cannot formulate problems but only follow detailed logical instructions. The computer is no substitute for intelligence. It is unlikely, for example, that Gauss could have reached his conclusions on normal distribution simply by feeding large amounts of data to a computer and waiting for something to appear. Competent, fundamentally-trained persons are, as always, uniquely necessary to important research, but their efforts can be advanced immeasurably by the modern computer. They need not be computer experts, but should have more than passing awareness of computer capabilities and should seek opportunities for personally applying their problems to the computer. One danger must be mentioned: As J. C. Hetrick has noted,[5] "a common reaction when computers are available is to

5 "Panel Discussion on Manpower Needs and Educational Programs," by J. C. Hetrick, Wayne Univ. Conference, as noted Ref. 1, p. 32.

assume that all problems, and even all sections of all problems, must be solved electronically." In our present passion for computers, there is an undesirable tendency for persons skilled in machine computation to look down upon their colleagues who have no such experience. This is an unfortunate attitude. It must be remembered that there will always be those persons who can make significant contributions irrespective of current fashions in scientific tools.

With the potentialities of the machine, and with the imagination and discrimination of the fundamentally-educated engineer, there is little doubt that the computer can be a tremendous stimulant to both research and its allied teaching program.

RELATIONS WITH ENGINEERING PRACTICE

The University, in addition to adjusting its curricular and research programs, must look beyond these bounds to its relationship with engineering practice. The University should stand ready to help solve those computer problems for which consulting practice and industry are not equipped, although it is not explicitly clear what form this help will take. Certainly service courses, clinics, and seminars sponsored by the University can be of major benefit to both industry and the University itself, especially as they allow interchange of ideas. There should be a strong spirit of geniality on the part of both, and on the University's part a readiness to communicate new ideas, with suggestions for their practical application.

However, the relationship between the University and engineering practice is not entirely one-sided. Industry, in its search for employees, is already beginning to place unwarranted emphasis on proficiency of graduates in computer design and use. It would be unwise for the University to give way to this clamor for increased quality and quantity of users and designers by instituting formal vocational courses in computer programming, coding, operation, and hardware building. It should insist that industry assume its fair share of training of computer personnel, especially in those areas which may be regarded as job-training, and for which industry is best equipped to give.

There is another point at which the University must stand guard. While most of the basic research leading to changes in design specifications originates within the University, practicing engineers have, and rightly so, a stronger voice in the final adoption of specifications. Practicing engineers have a large monetary investment in computer programs for design procedures based on current specifications. There are well-founded fears that they may be exceedingly reluctant to see any changes that would devalue or eliminate those programs and require extensive revisions, the more so as their competitive position is reduced. There is already enough inertia in translating research results into good design practice, and engineers in both the University and everyday practice must cooperate to avoid this possible deterrent to progress. Many traditional methods of analysis and design based on overly-simplified approximations will be gradually discarded in spite of a natural hesitancy to give up familiar procedures. In the words of Nathan Newmark,[6] F. ASCE,

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[6] "A Revolution in Design Practice," by N. M. Newmark, Civil Engineering, May, 1958.

> "We tend [now] to use computers, because of the gain in speed or productivity, along with methods that are outmoded or at least inaccurate and undependable. . . . The next step is the development and use of more accurate analytical procedures. But we must look forward eventually to the development of more fully automated design techniques. This implies more sophisticated design criteria and more rationally based design specifications. When that time comes it will certainly be the end of the handbook engineer."

Overly-simplified methods have grown out of a desire to reduce the demands on the time and ability of the designer, but this is no longer a tenable reason for retaining them in view of the speed and refinement possible with computers. In fact, the increased computational ability with modern computers provides the University with considerable incentive to teach advanced analytical methods. The receptiveness of the profession to such refined and better methods will be hastened largely by the efforts of the University in providing fundamental technical education and in producing, thereby, competent and imaginative engineers spontaneously receptive to new ideas.

OTHER UNIVERSITY AREAS

Other areas within the University also have felt the influence of the computer. While it is not the purpose of this paper to discuss the influence of computers on fields other than engineering, brief recognition must be given to the support and nourishment which engineering stands to receive from this source. Engineers cannot afford to overlook or fail to encourage the recent advances and prospects in machine translation of languages, and in automated search and retrieval of library documents, both of which will spur research by easing particularly troublesome tasks. Of major importance is the recent stimulation of applied mathematics, and especially numerical analysis. It was not so long ago that the very term "numerical analysis" was a profanity in mathematical circles, but now it is quite an acceptable endeavor for a mathematician. Important basic research in applied mathematics is under way. Much attention has already been given to developing the laws which govern numerical analysis, since the use of discrete numbers throws out the traditional rules of algebra. Problems such as ill-condition, the distribution and propagation of errors, choice of numerical method, and choice of interval are now recognized by mathematicians as legitimate and intellectually satisfying areas for investigation. There has grown up a healthy ebb and flow of ideas between engineering and applied mathematics, and in a sense, this recent mathematical activity is a return blessing, since many of the problems first appeared in engineers' efforts at computer solution. In short, engineering education stands to gain many benefits from this interchange of computer ideas and experiences with other areas of the University.

COMPUTER CENTER

One of the really effective media within the University for exchange of ideas and abilities among the staff is the computer center, which serves to bring together people of a wide range of interests. It is therefore pertinent to examine its role in supporting the University's general educational efforts.

It is widely agreed that a single moderately large computing facility accessible to all is preferable to several small facilities duplicated and maintained by autonomous departments within the University. The computer should be easily available to personnel from all University departments, not only to professors and graduate students in carrying out their researches, but also to undergraduates as their interest and ability allows and to neighboring industries which need help in computing problems. The University computer center should be strictly educational in nature, helping users and providing consultation where necessary, but accepting no fundamental responsibility for solving problems without direct participation of the user. It is the user, in most instances, who knows most clearly the physical aspects of his problem and is the best judge of its progress and satisfaction in solution. Full participation by the user may not be expeditious for large-scale research programs or when demands on machine time approach capacity, but the computer center should never be like a soft-drink machine where the consumer puts in his dime and takes out a bottle. Such a purely accommodating service operation is not education nor will it stimulate new ideas. The delivery of computing service must be secondary to academic objectives and, consistent with this concept, the University computing center should never encourage contracts for outside computing jobs. Likewise, it is unfair and inconsistent for a University administration to insist that the computing center be self-supporting, any more than the library should be, since there will always be a consequent inclination to schedule the paid work to the detriment of the equally important unpaid. The staffing and operation of a computer center and periodic acquisition of improved machines raise serious financial problems beyond the scope of this paper. No university can treat with indifference an initial expenditure of a half-million dollars and an annual budget of $50,000. Nevertheless, any financial arrangement must be consistent with the center's function as a laboratory to support the formal curriculum and the research necessary to invigorate the teaching staff. It is most important that a dominant educational role of the computer center always be fostered.

CONCLUSIONS

It has been interesting to speculate on the influence which digital computers are having and will have on engineering education. Even these remarks serve to show that it is one of the most generally significant revolutions ever faced by engineers. In spite of the problems, financial, operational, and philosophical, there emerges a pattern of invigoration, of enthusiasm, of willingness to revise traditional thinking, of increased research potential, of deep and renewed appreciation of fundamentals. Whether or not the suppositions expressed herein stand the test of time, few persons would deny that engineering education is being impelled into an era of genuine enrichment.

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DISTRIBUTIONS OF EXTREME WINDS IN THE UNITED STATES[a]

By H. C. S. Thom[1]

SYNOPSIS

Following previous work, annual extreme wind speed data for airports in the United States were corrected for anemometer type and reduced to a standard 30-ft elevation. The Fisher-Tippett Type II distribution was fitted, giving 0.50, 0.02, and 0.01 quantiles. Confidence intervals of 0.90 for the quantile isolines and a nomogram for conversion of quantiles to desired elevations are developed.

INTRODUCTION

In a previous paper[2] the best available design variable, the annual extreme-mile wind speed, was chosen on the basis of a rational consideration of the requirements for wind-speed values in structural design. The extreme-mile wind speed is defined as the 1-mile passage of wind with the highest speed for a day. This led to the basic-data series of annual extreme-fastest-mile wind speeds which were found to lack homogeneity because of changes in anemometer elevations. This difficulty was overcome by reducing the annual extreme wind speeds by Hellmann's one seventh power law, which was found to be suitable for such conversion by P. O. Huss.[3] The conversion was made to a stand-

Note.—Discussion open until September 1, 1960. To extend the closing date one month, a written request must be filed with the Executive Secretary, ASCE. This paper is part of the copyrighted Journal of the Structural Division, Proceedings of the American Society of Civil Engineers, Vol. 86, No. ST 4, April, 1960.

[a] This study was supported, in part, by the Bureau of Public Roads, United States Department of Commerce.

[1] Chf. Climatologist, Office of Climatology, U. S. Weather Bur.

[2] "Frequency of Maximum Wind Speeds," by H. G. S. Thom, Proceedings, ASCE. Vol. 80, 1954, 11 pp.

[3] "Relation between Gusts and Average Wind Speeds," by P. O. Huss, Daniel Guggenheim Airship Inst., Akron, Ohio, Report No. 140, 1946.

ard elevation of 30 ft as suggested by the Structures Committee. This gave a data series suitable for climatological analysis.

The Fisher-Tippett extreme-value distribution theory was reviewed and their Type II distribution was chosen to fit the annual extreme wind series. A probability paper for the distribution was developed and a method of plotting was shown. The method of maximum likelihood was applied to estimate the parameters β and γ of the Type II distribution function

$$F(x) = \exp\left[-(x/\beta)^{-\gamma}\right] \quad \ldots\ldots\ldots\ldots\ldots\ldots (1)$$

In Eq. 1 F(x) is the probability of a wind speed being less than x. The mean recurrence interval R is given by the relationship

$$R = 1/[1-F(x)] \quad \ldots\ldots\ldots\ldots\ldots\ldots (2)$$

The differential equations for estimating β and γ were found to be

$$\frac{\partial L}{\partial \beta} = \frac{n}{\hat{\beta}} - \hat{\beta}^{\hat{\gamma}-1} \sum_{i=1}^{n} x_i^{-\hat{\gamma}} = 0 \quad \ldots\ldots\ldots\ldots\ldots\ldots (3)$$

and

$$\frac{\partial L}{\partial \gamma} = \frac{n}{\hat{\gamma}} - \sum_{i=1}^{n} \log x_i + \frac{n \sum_{i=1}^{n} x^{-\gamma} \log x_i}{\sum_{i=1}^{n} x_i^{-\hat{\gamma}}} = 0 \quad \ldots\ldots\ldots\ldots (4)$$

It was shown how to solve these equations graphically. This method is laborious and satisfactory only for fitting one or a few distributions. Large-scale fitting problems such as the one of concern here must be accomplished with the aid of electronic digital computers. The results of fitting for mixed City Office and Airport Office data at Fort Wayne, Ind. are available.[2]

DATA REDUCTION

Shortly after the 1954 paper[2] was completed, discussions with George J. Malloy[b] suggested that the wind regimes in the open country (airport stations) might be quite different than downtown locations (city stations). A study of Boston, (Mass.) city office and airport data showed that the probability of exceeding 60 mph at the airport was twice that at the city office. This surprising discovery threw doubt on the validity of previously furnished climatological series used in several studies;[4,5,6,7] however, there were no facilities avail-

[b] Chf. Engr., Travelers Fire Insurance Co.

[4] "American Standard Building Code Requirements for Minimum Design Loads in Buildings and Other Structures," by Amer. Standards Assn., A58.1-1955, Approved September 3, 1955.

[5] "Wind Forces on Structures: Nature of Wind," by R. H. Sherlock, Proceedings, ASCE, Vol. 84, No. ST 4, July, 1958.

[6] "Wind Pressures in Various Areas of the United States," by G. N. Brekke, Natl. Bur. of Standards, Bldg. Materials and Structures Report 152, 1959.

[7] "Statistics of Extremes," by E. J. Gumbel, Columbis Univ. Press, New York, 1958.

able at the time to study the matter further. Recently the Bridge Division of the Bureau of Public Roads, United States Department of Commerce, (BPR) provided the support necessary to make the present study.

The present analysis has the advantage over previous work of an additional 8 or 9 yr of records at airport stations. These were processed separately from city office data. Corrections were made for anemometer type where necessary, and all data series were reduced to a standard 30-ft elevation by Hellman's law. The city office data were processed in a similar manner for comparison purposes. Altogether 131 airport station records and 165 city station records were processed. Ten city station records, near large bodies of water, were included with the airport records in order to adequately reflect coastal conditions. This made a total of 141 open-country stations with about 1,700 yr of record averaging about 15 yr per station. As will be seen later the isotachs of wind are more reliable than the average record length might indicate because several stations contribute to the establishment of an isotach.

The 0.02 quantiles (wind speed for which the probability of being exceeded is 0.02) for city office versus airport stations are shown in Fig. 1. Although the 0.02 quantiles at city offices exceed the airport values at a few stations, probably because of exposure or random sampling, the preponderance of airport wind speeds are considerably greater than those for city offices. In fact, the weighted average of this quantile for airports is 74.8 mph as contrasted to the city offices weighted average of 55.3 mph. Thus at high quantiles, airport speeds average about 20 mph faster than city offices. This is sufficient reason not only to strictly avoid mixing airport and city observations, but also to use the airport data as a measure of open-country conditions. Extreme wind speeds are assumed to come from any direction unless local conditions dictate otherwise. Special conditions of exposure will be cited subsequently.

CLIMATOLOGICAL ANALYSIS

A review of the literature on extreme value distributions published since 1954, showed nothing which would change our method of analysis. E. J. Gumbel pointed out[7] a fact which was not explicitly mentioned[2] by the writer, although Eq. 5, to be presented subsequently, and the logarithmic scale on the abscissa Fig. 3 of the 1954 paper[2] should have made it clear. This is that the Type II distribution may be transformed to a Type I by a logarithmic transformation. The fitting of logarithms of the wind speed was tried with J. Lieblein's estimation procedure,[8] but the results were not satisfactory because the method, at present, does not have the necessary tables to handle large subsamples. A very excellent and exhaustive discussion of extreme-value distribution theory is now available.[7]

L. E. Anapol'skaia and L. S. Gandin,[9] using the writer's 1954 paper[2] as a basis, rederived the moments of the Type II distribution which were already well known, and then applied these to fitting the distributions to Russian stations. The efficiency of the moment estimate for β is poor, and its use results, in effect, in discarding about 40% of the observations, a very undesirable practice with scarce meteorological data. Presumably they could not use the more

[8] "A Method of Analyzing Extreme Value Data," Natl. Advisory Comm. for Aero., Washington, Tech. Note 3053, 1954.

[9] "Metodika Opredeleniia Raschetnykh Skorostei Vetra dlia Proektirovaniia Vetrovykh Nagruzok na Stroitel'nye Sooruzheniia," by L. E. Anapol'skaia and L. S. Gandin, Meteorologiia i Gidrologiia, No. 10, 1958, pp. 9-17.

efficient method of maximum likelihood because of a lack of electronic digital computer facilities. The Russian authors did not indicate familiarity with rather extensive work in the field of extreme-value theory of the well-known Russian mathematician B. V. Gnedenko.

Since near-optimum methods of fitting were already available,[2] Eqs. 3 and 4 were used in fitting Fisher-Tippett Type II frequency curves to the climatological series from the 141 open-country stations. This was accomplished using the IBM 650 electronic computer in an iteration procedure which gave the

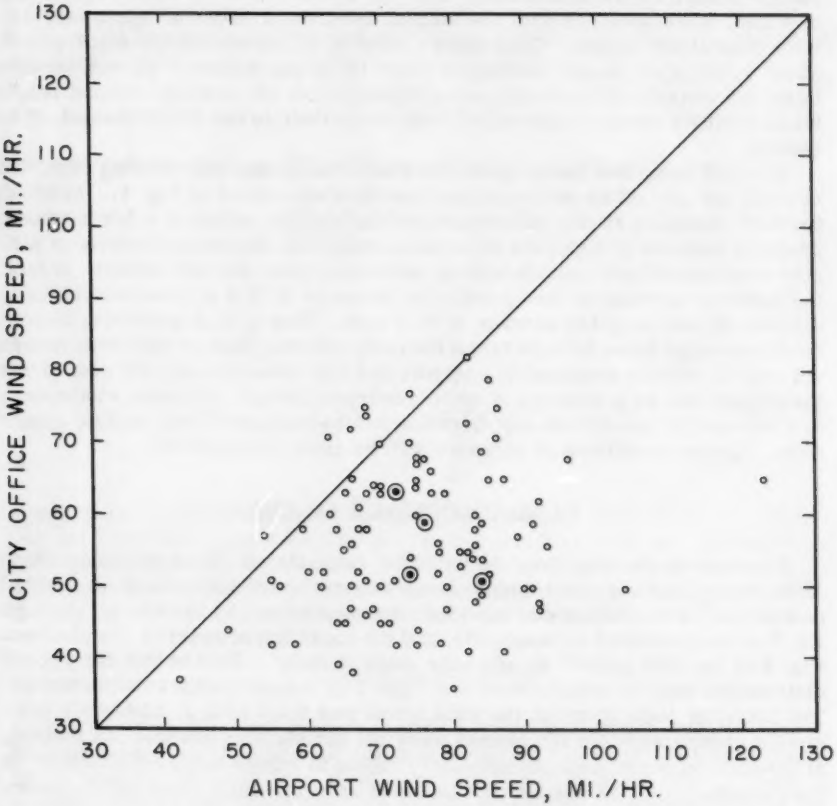


FIG. 1.—CITY OFFICE VS. AIRPORT FOR 0.02 WIND-SPEED QUANTILES

β and γ roots of Eqs. 3 and 4 to a prescribed accuracy. The computer was used on these to compute a series of quantiles among which were the 0.01 (100 yr), 0.02 (50 yr), and the 0.50 quantiles (medians) (2 yr mean recurrence interval winds). These were plotted on maps and analyzed with isolines whose intervals are suitable for conversion to pressure. These are shown in Figs. 2, 3, and 4. The particular quantiles were chosen because it was thought that the 50-yr and 100-yr values might be directly suitable for many design problems

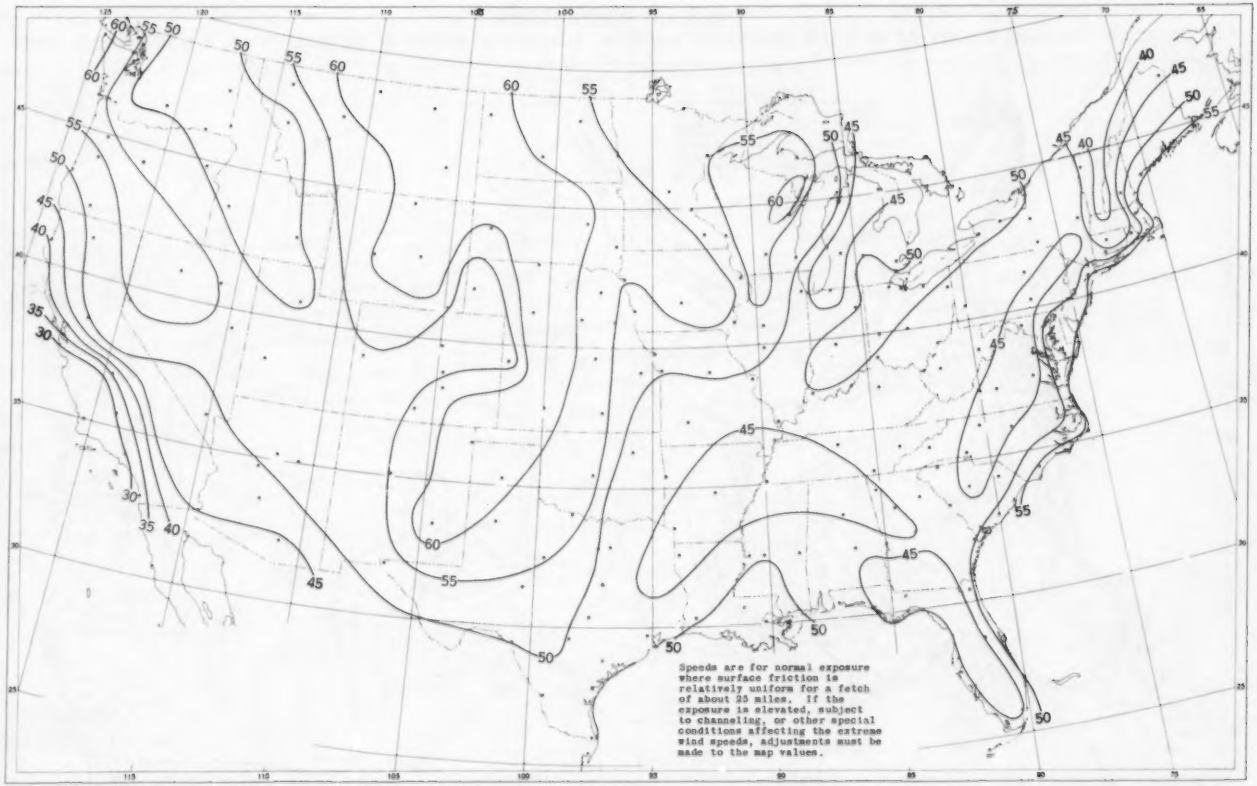

Speeds are for normal exposure where surface friction is relatively uniform for a fetch of about 25 miles. If the exposure is elevated, subject to channeling, or other special conditions affecting the extreme wind speeds, adjustments must be made to the map values.

FIG. 2.—ISOTACH 0.50 QUANTILES, IN MILES PER HOUR. ANNUAL EXTREME-MILE 30 FT ABOVE GROUND, 2-YR MEAN RECURRENCE INTERVAL.

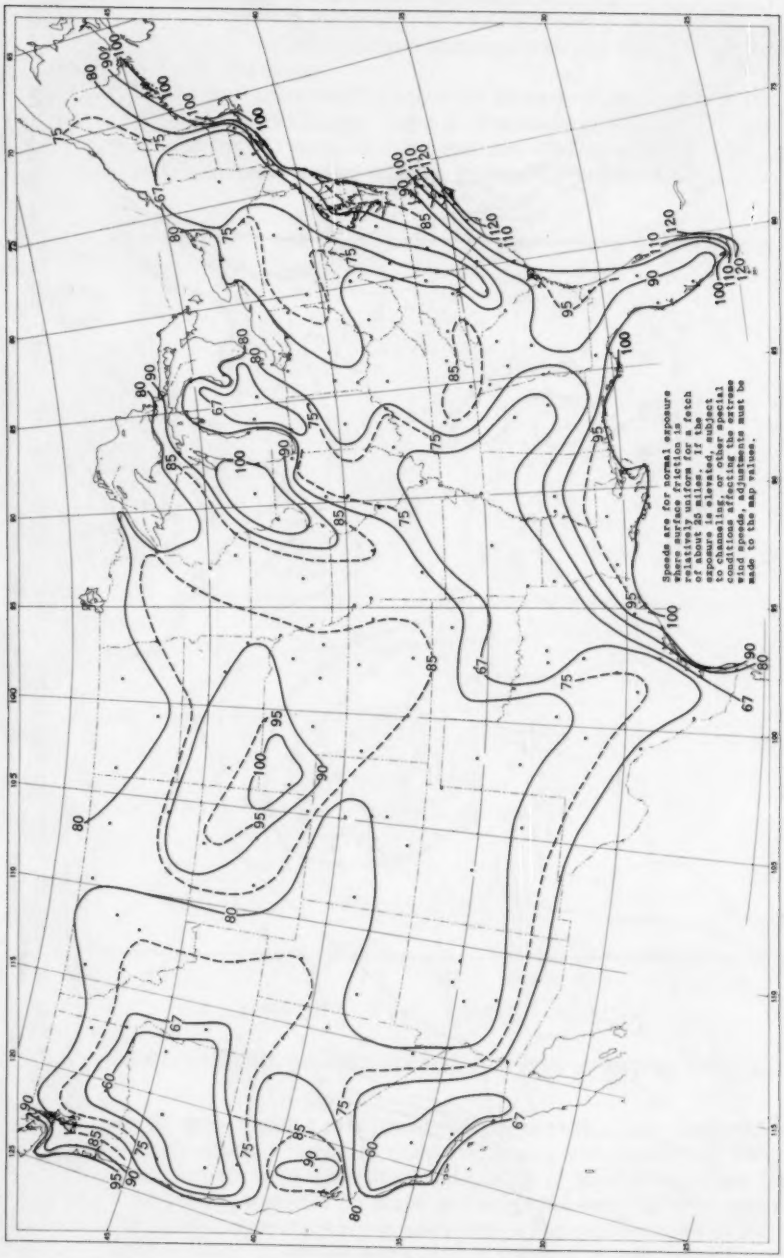

Speeds are for normal exposure where surface friction is relatively uniform for a fetch of about 10 miles. If the terrain is irregular, subject to channeling, or other special conditions affecting the extreme wind speeds, adjustments must be made to the map values.

FIG. 3.—ISOTACH 0.02 QUANTILES, IN MILES PER HOUR. ANNUAL EXTREME-MILE 30 FT ABOVE GROUND, 50-YR MEAN RECURRENCE INTERVAL.

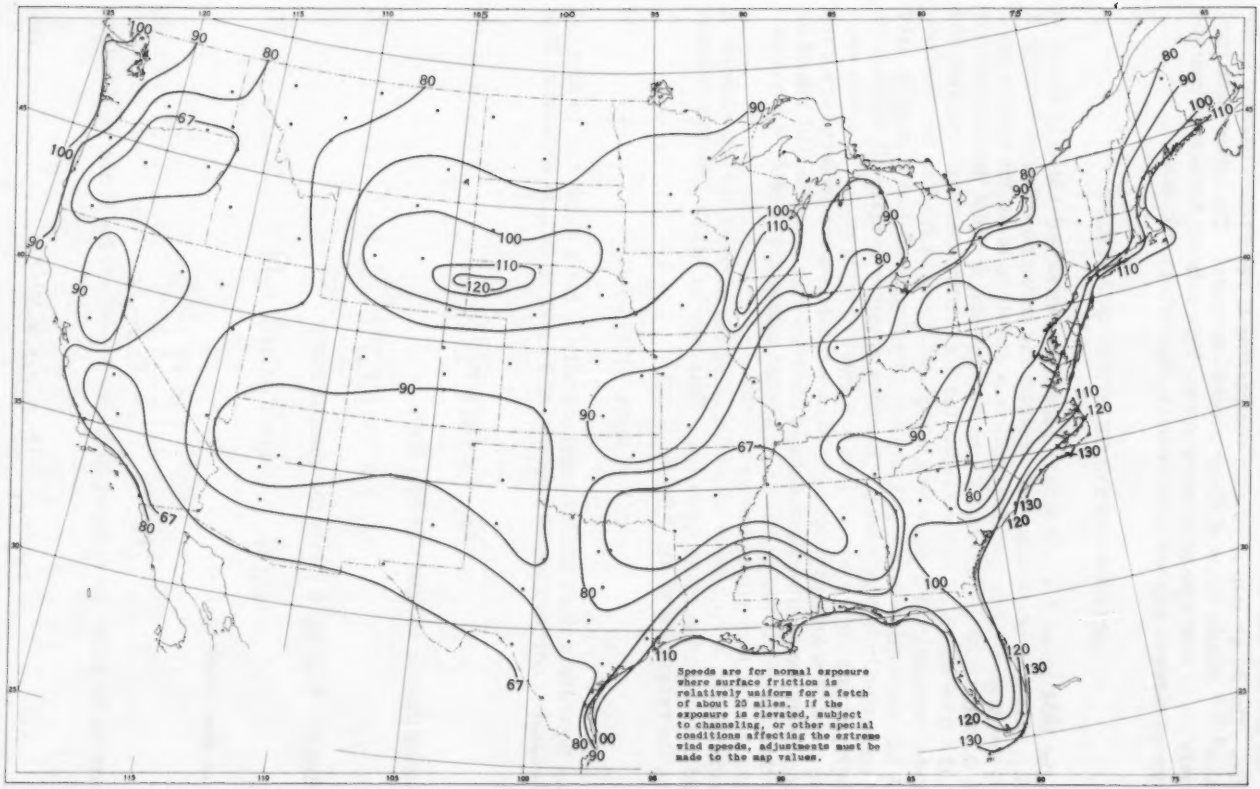

Speeds are for normal exposure where surface friction is relatively uniform for a fetch of about 25 miles. If the exposure is elevated, subject to channeling, or other special conditions affecting the extreme wind speeds, adjustments must be made to the map values.

FIG. 4.—ISOTACH 0.01 QUANTILES, IN MILES PER HOUR. ANNUAL EXTREME-MILE 30 FT ABOVE GROUND, 100-YR MEAN RECURRENCE INTERVAL.

while the median was added to provide the means for obtaining any desired quantile. The 50-yr map is roughly comparable in recurrence interval to the available[4,5,6] simple maps of 40-yr record extremes. The latter show considerably lower wind speeds in many areas. This reflects further the marked differences between city and open-country, high-wind conditions.

DETERMINATION OF OTHER QUANTILES

In the 1954 paper,[2] a probability paper was developed which is shown in Fig. 5. This may be readily used to obtain the entire distribution function F for any location on the maps. The process is simply to locate the desired point on the 0.50 and 0.02 maps, read off the wind speeds, and plot them at 0.50 and 1 - 0.02 = 0.98 on Fig. 5. A straight line connecting these points gives the complete distribution F from which any quantile may be read on the abscissa. Fig. 5 has been deliberately left without an example so that it may be used more conveniently by the reader. It was impossible to make all three maps absolutely consistent, however, they are close to being so. A check on estimating the 0.01 quantile by this method, at a considerable number of arbitrary latitude-longitude intersections, showed discrepancies from the 0.01 map of no more than 5 mph. If 100-yr, 50-yr or median values are needed, it is recommended that the map values always be used.

Occasionally, a procedure which holds on the average for all stations is needed for obtaining other quantiles. This may be easily found as follows: Taking the reciprocal of Eq. 1 gives

$$1/F = \exp(x/\beta)^{-\gamma} \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad (5)$$

Taking logarithms twice (double logs) and solving for x which may be labelled x_{F_1} because it corresponds to a particular F_1 and, in fact, is a quantile. This yields

$$x_{F_1} = \beta/\log(1/F_1)^{1/\gamma} \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad (6)$$

Repeating this for the quantile x_{F_2} gives

$$x_{F_2} = \beta/\log(1/F_2)^{1/\gamma} \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad (7)$$

Dividing Eq. 7 by Eq. 6 yields the quantile ratio

$$x_{F_2}/x_{F_1} = [\log(1/F_1)/\log(1/F_2)]^{1/\gamma} \quad \ldots\ldots\ldots \quad (8)$$

This is equivalent to

$$x_{F_2}/x_{F_1} = \left[\frac{\ln F_1}{\ln F_2}\right]^{1/\gamma} \quad \ldots\ldots\ldots\ldots \quad (9)$$

because the negative signs cancel. Since $\log F \sim -(1-F)$ for F near 1, it follows that

$$x_{F_2}/x_{F_1} \sim [(1-F_1)/(1-F_2)]^{1/\gamma} \quad \ldots\ldots\ldots\ldots \quad (10)$$

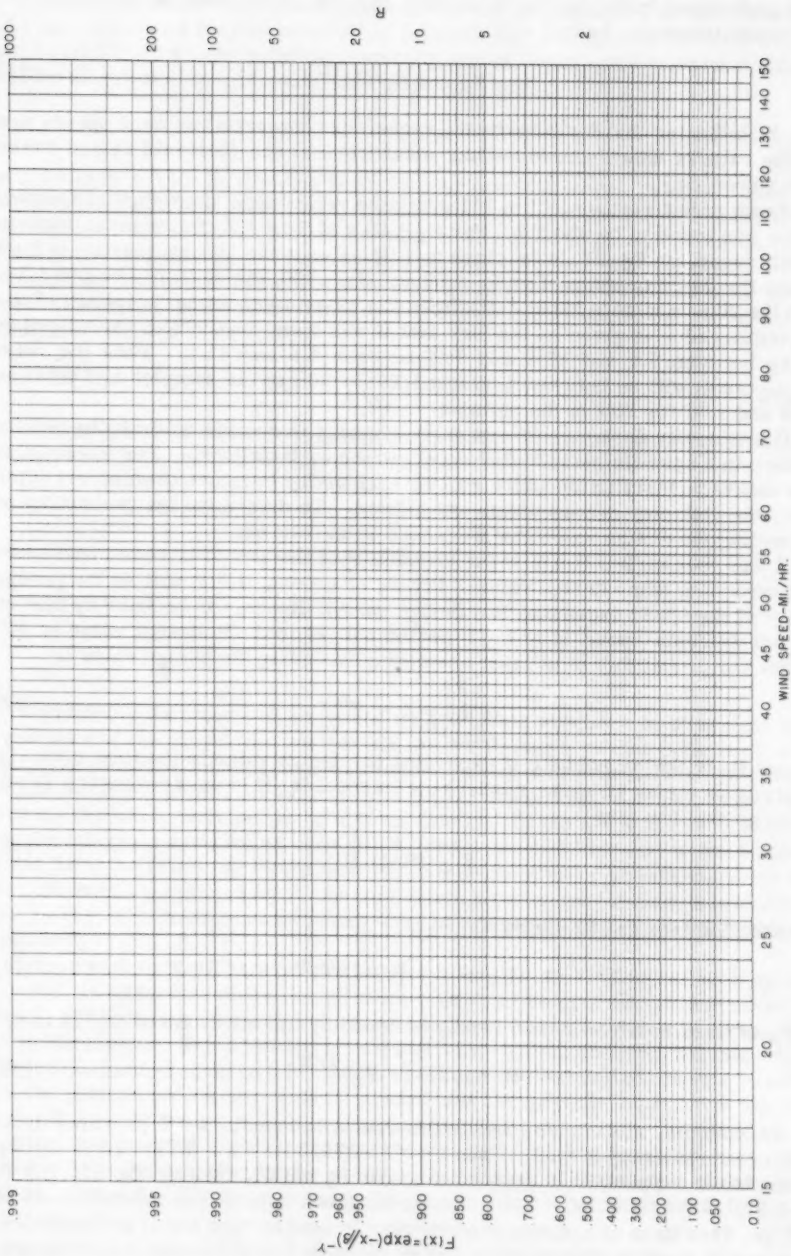


FIG. 5.—MAXIMUM-VALUE PROBABILITY PAPER, FISHER-TIPPETT TYPE II DISTRIBUTION.

The right-hand term, by Eq. 2, is seen to be the reciprocal of the mean recurrence intervals, hence

$$x_{R_2}/x_{R_1} \sim (R_2/R_1)^{1/\gamma} \quad \ldots\ldots\ldots\ldots\ldots\ldots (11)$$

Eq. 11 indicates that for large mean recurrence intervals two wind speeds are in the ratio of their corresponding mean recurrence intervals raised to the $1/\gamma$ power.

From an examination of the 141 available values of γ, the weighted average value was found to be about 9. For $R_1 = 50$ and $R_2 = 100$, the quantile ratio is easily found, by Eq. 11, to be close to 1.08. Hence the 100-yr quantile is 1.08 times the 50-yr quantile. For $R_1 = 50$ and $R_2 = 200$, the ratio is close to 1.16, and the 200-yr wind is about 1.16 times the 50-yr wind. Some indication of the variability of γ is given by the fact that it was found that 90% of the values of γ lay between 5.5 and 15.5 and 70% between 6.5 and 11.5. Thus the ratio x_{F100}/x_{F50} will be between 1.05 and 1.13 for 90% of the samples and between 1.06 and 1.11 for 70% of the stations.

Occasionally design problems arise where it is desired to limit the risk of failure in a specific period after construction or installation. For example, if one makes an installation which will be removed or becomes obsolescent after 3 yr, the risk only occurs within this period. To determine the probability of exceeding the design value the procedure is as follows:

It was seen earlier that F is the probability of the annual extreme wind speed not exceeding some design value. (Then, of course, 1 - F will be the probability of the speed exceeding that design value). Hence, the probability that no annual extreme speed in an n-yr period will exceed the design value is F^n. Hence the probability of at least one speed exceeding the design value is

$$P_n = 1 - F^n \quad \ldots\ldots\ldots\ldots\ldots\ldots (12)$$

where P_n is the probability of exceeding the design value in the first n yr. To express in terms of mean recurrence interval R, it is only necessary to rearrange Eq. 2 to give

$$F = (R-1)/R. \quad \ldots\ldots\ldots\ldots\ldots\ldots (13)$$

Substitution into Eq. 12 yields

$$P_n = 1 - [(R-1)/R]^n \quad \ldots\ldots\ldots\ldots\ldots\ldots (14)$$

If P_n is fixed in advance then R may be found by the inversion of Eq. 14 to be

$$R = 1/[1-(1-P_n)^{1/n}] \ldots\ldots\ldots\ldots\ldots\ldots (15)$$

As an example, assume an installation that is to remain n = 3 yr with a mean recurrence interval R = 50. Then $P_3 = 1 - (49/50)^3 = 1 - 0.94 = 0.06$. On the other hand, suppose it is desired to make $P_3 = 0.03$, then by Eq. 15, $R = 1/(1-0.97)^{1/3} = 1/(1-0.99) = 100$. Hence, the mean recurrence interval must be 100 yr. This form of analysis would appear to lead to very small probabilities; however, it must be remembered that P_n considers only the wind speed exceeding the design value and does not consider the probabilities of other factors in a design, which, if included, give quite different results.

Approximate 0.90 confidence intervals for the probabilities of the isotachs on Figs. 2, 3, and 4 may be obtained as follows: For a single station with average record of 15 yr, the relative frequency u/n of winds below a given quantile will be distributed in a binomial distribution

$$f(u) = \binom{15}{u} (1-F)^u \; F^{15-u} \quad \ldots\ldots\ldots\ldots\ldots (20)$$

on a scale of u/15. In drawing isotachs four stations were, in effect, used to define the isotach at a given point, that is, drawing proceeded by groups of four stations in order to obtain proper curvature. Hence, it is assumed that an isotach, at any point, is approximately as accurate as a four-station mean. The correlation between extreme winds in such groups of four was found to be near zero. Consequently, the four values of u are distributed independently and their mean u/60 has the distribution

$$f(u) = \binom{60}{u} (1-F)^u \; F^{60-u} \quad \ldots\ldots\ldots\ldots\ldots (21)$$

on a scale of u/60. The 0.90 confidence interval for the estimated probability F may be found by summing Eq. 21 such that

$$0.05 = \sum_{u=0}^{u=u_{0.05}} f(u) \quad \ldots\ldots\ldots\ldots\ldots (22)$$

and

$$0.95 = \sum_{u=u_{0.95}}^{u=n} f(u) \quad \ldots\ldots\ldots\ldots\ldots (23)$$

in determining the values of $u_{0.05}$ and $u_{0.95}$. This was done by graphical interpolation of the binomial distributions for n = 60 and F = 0.50, 0.98, and 0.99. This gave the general confidence value $u_F/60$, which will be somewhat too broad for the binomial gives nonparametric limits. These must be used, however, because the exact distribution of u is not known. In the above order, the confidence intervals were found to be (0.39, 0.61), (0.94, 0.995) and (0.96, 0.998). These have the following interpretation: If π is the true or population value of the probability estimated by F, the confidence interval will include π with 0.90 probability. The confidence intervals may be converted to u_{1-F} by subtracting from unity.

Conversion to confidence values for the quantiles, as read from the Figs. 2, 3, and 4, is somewhat more complicated. One of two procedures may be followed: The distribution curve F, for a particular location, may be drawn on the graph paper of Fig. 5 using the 0.50 and 0.02 values read from the maps for that location, as discussed previously. Then the probability confidence values for the desired probability may be plotted and the quantile confidence limits read off directly from the F-line in miles per hour.

The other method of obtaining confidence limits for the quantiles is to find β and γ from the quantiles $x_{0.50}$ and $x_{0.02}$ read from the maps at a chosen location. From Eq. 8 it follows immediately by taking logarithms that

$$\gamma = 1n[1n(1/0.50)/1n(1/0.98)] / 1n(x_{0.98}/x_{0.50})$$

$$= 1.25276/1n(x_{0.98}/x_{0.50}) \quad \ldots\ldots\ldots\ldots\ldots (24)$$

Now for a particular quantile it follows from Eq. 7 that

$$\ln \beta = \ln x_F + (1/\gamma) \ln \ln (1/F) \quad \ldots\ldots\ldots\ldots (25)$$

To obtain β, the median value (F = 0.50) may be always used in Eq. 25 giving

$$\ln \beta = \ln x_{0.50} + 0.69315(1/\gamma) \quad \ldots\ldots\ldots\ldots (26)$$

Hence to obtain a chosen quantile x_F knowing β and γ from Eqs. 24 and 26, it is only necessary to solve Eq. 25 for $\ln x_F$ giving

$$\ln x_F = \ln \beta - (1/\gamma) \ln \ln (1/F) \quad \ldots\ldots\ldots\ldots (27)$$

The quantile confidence limit is then obtained from Eq. 27 by substituting β and γ and the corresponding probability confidence limits for F.

For the average β, 45, and the average γ, 9, substitution of the F intervals into Eq. 27 gives the 0.90 confidence intervals for the 0.50, 0.02, and 0.01 quantiles (2 yr, 50 yr, and 100 yr mean recurrence intervals) as (45, 49), (61, 81), and (64, 90) respectively, all in miles per hour.

WIND CLIMATE AND ELEVATION CONVERSION

For present purposes extreme winds may be the result of three meteorological situations: extratropical cyclones, tropical cyclones, and thunderstorms. Tornadoes have not been considered in this study because there are no valid observations of their extreme speeds. Furthermore, the probability of a tornado striking a single structure is extremely small. Extratropical cyclones generally produce high winds during the winter months because they are energized by the temperature contrasts between air masses. Such contrasts tend to be larger during the colder months. Thunderstorms and tropical cyclones are both convectionally driven and hence produce high winds during the summer months. The West Coast, Great Lakes region, and mountainous areas usually have their extreme winds during the colder months due to extratropical cyclones. The Gulf Coast and Atlantic Coast, as far north as Southern Maine, have experienced most extreme winds from tropical cyclones. The effect of these storms often extends inland from 100 to 200 miles. The remainder of the United States experiences extreme winds largely from thunderstorms. This type of storm accounts for over one third of the extreme-wind situations in the United States proper.

The isotach quantiles shown on the maps of Figs. 2, 3, and 4 are for normal conditions of exposure only. Such conditions prevail where the surface friction is relatively uniform for a fetch of about 25 miles. If the exposure is elevated, subject to channeling of the wind, or other unusual or special conditions affecting wind speed, suitable adjustments to the map values must be made.

Water areas have a marked effect on extreme-wind speeds. Where a location has unobstructed access to a large body of water, extreme winds may be 30 mph or more greater than a short distance inland. The isotachs near large-water bodies in Figs. 2, 3, and 4 have been drawn to reflect, as much as possible, this effect of water. High winds in cyclones and near water tend to remain steady over longer periods than for thunderstorms. They also tend to be much more widespread in a given situation and hence cause widespread damage, al-

though damage to individual structures may not be greater than for thunderstorm winds.

Gust factors and pressure conversion will not be discussed in detail herein since they have been very adequately treated.[6] Based on the work of Huss[3] and others, a standard gust factor at 30 ft was taken as 1.3 by G. N. Brekke.[6] This or other gust factors may be multiplied by the wind speeds shown on Figs. 2, 3, and 4 to give gust speeds. When the wind speed V is in miles per hour, the velocity pressure in pounds per square foot, given by Brekke,[6] is $0.00256\ V^2$ for normal sea-level conditions. This applies over the greater part of the United States, since elevation differences are not sufficient to affect the constant greatly.

Since the increase of speed with height depends in the first 1,000 ft mainly on surface friction, the type of storm situation could make little difference. Huss[3] found that the one-seventh power law gave good approximations up to 350 ft over land for the fastest mile for all quantiles having probabilities less than 0.90. There is no evidence to indicate that it does not hold to higher levels over land so it has been assumed here that it is valid up to 850 ft. The Hellman formula, with one-seventh power, is

$$V_z = V_{30}(z/30)^{1/7} \quad \ldots\ldots\ldots\ldots\ldots\ldots (28)$$

Where z is the desired elevation in feet, V_{30} is wind-speed quantile at 30 ft in miles per hour, and V_z is the converted speed at elevation z. This equation has been solved in the nomogram of Fig. 6. The nomogram may be easily understood by following the dotted lines from z = 215 ft to the curve, thence to the V_z/V_{30} scale, thence to V_{30} = 96 mph, and finally to the V_z scale to find the speed at 215 ft, V_{215} = 128 mph. The nomogram may also be used in reverse to convert winds at higher levels to the 30-ft elevation. If this step is followed by a forward conversion step, the nomogram gives a means of converting from a given elevation to any other elevation. The nomogram should not be used for elevations below 30 ft as the one-seventh power law is of doubtful validity below this elevation.

ACKNOWLEDGMENTS

The writer is particularly indebted to H. N. Wilcox, of the Bridge Division, BPR, who recognized the need for and promoted this study, and who, through many technical discussions, helped guide the study into its present form. Sincere thanks must also be extended to the meteorologists of some 300 first-order Weather Bureau Stations, who took time from their busy schedules to tabulate the basic data. Gladys Crown and Maurice Kasinoff processed these data to final form. The latter also did all manual computations and worked the model distribution problem for checking the electronic computer. G. L. Barger, acting director of the National Weather Records Center at Asheville, N. C., aided immeasurably by expediting the collection of the data and performance of card punching and electronic-computer work. Especial thanks are due to R. L. Joiner and J. R. Roddy for preparing the IBM 650 program and to Grady McKay for running the computation project through under considerable pressure to get the job done. Sincere thanks are also offered to W. W. Buck, Jr., for his careful rendition of the illustrations.

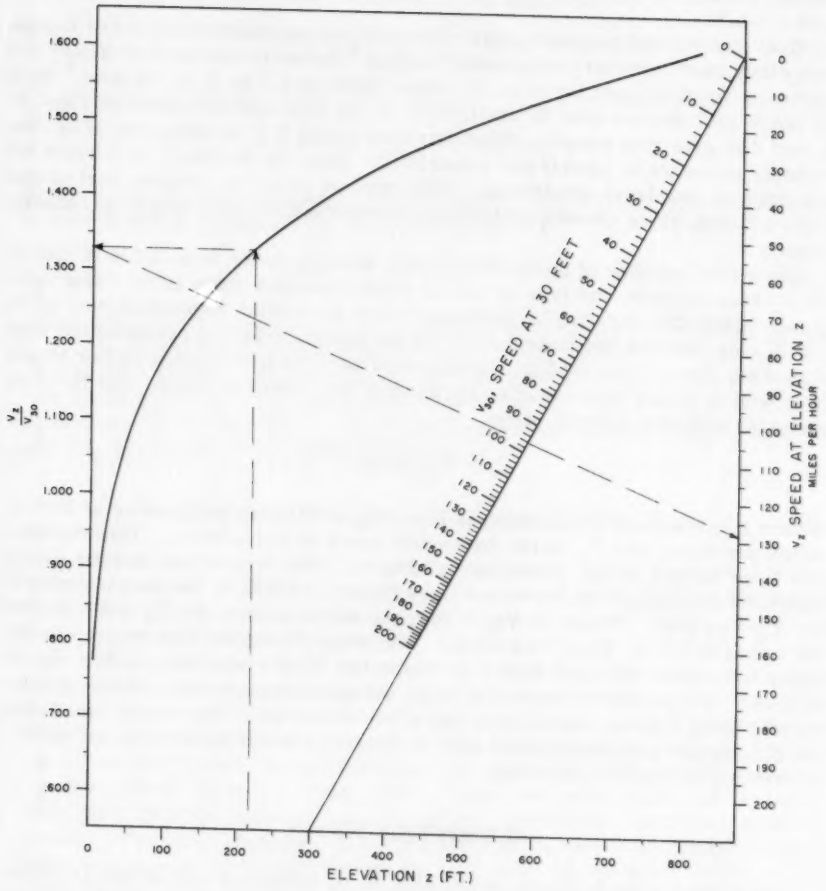


FIG. 6.—WIND SPEED-HEIGHT NOMOGRAM FOR SOLUTION OF EQ. 28.

Journal of the
STRUCTURAL DIVISION
Proceedings of the American Society of Civil Engineers

EFFECT OF FLOOR SYSTEMS ON PONY TRUSS BRIDGES

By Robert M. Barnoff,[1] M. ASCE and William G. Mooney[2]

SYNOPSIS

An analytical method for determining the buckling load of top chords of single-span pony truss bridges has been developed by Edward C. Holt, Jr., M. ASCE. Ten model bridges were tested for failure to supplement this analysis.

In Holt's studies, the effect of floor-system participation on the top-chord stresses was neglected in both the analytical and experimental work. The participation of the floor system relieves the stresses in the bottom chord and reduces the effective depth of the truss, thus increasing the top-chord stresses. This paper presents the results of an analytical and experimental study of the effect of floor-system participation on top-chord stresses in pony truss bridges.

For the experimental portion of the study, a floor system was added to the model bridge used in the buckling studies. This floor system consisted of four continuous stringers clamped to the top flange of the floor beams. The model bridge was tested at working loads with and without the floor system, and the results compared. This experimental work indicated that the top-chord stresses are increased by 2% or 3%, under normal working loads, due to the effect of floor-system participation.

An analytical method is developed to calculate the increase in top-chord stresses due to floor-system participation. The analytical and experimental results compare reasonably well.

Note.—Discussion open until September 1, 1960. To extend the closing date one month, a written request must be filed with the Executive Secretary, ASCE. This paper is part of the copyrighted Journal of the Structural Division, Proceedings of the American Society of Civil Engineers, Vol. 86, No. ST 4, April, 1960.

[1] Asst. Prof. of Civ. Engrg., The Pennsylvania State Univ., University Park, Pa.
[2] Asst. Editor, McGraw-Hill Pub. Co., New York.

Although the results of the tests apply only to a single-span pony truss bridge model with continuous stringers, these results can probably be extended to include full-scale bridges. It is concluded from the test results and from measurements made on actual structures that the effect of floor-system participation cannot be neglected.

INTRODUCTION

The purpose of this paper is to evaluate the effect of floor-system participation on the stresses in the top chords of uniformly loaded, simple span pony truss bridges. This work is an extension of the pony truss project begun by Edward C. Holt, Jr., at The Pennsylvania State University. Holt's work dealt with the buckling problem of the top chord as a continuous elastically-restrained beam-column and the evaluation of bending moments in the truss members resulting from the deflection of the transverse frames.

The effect of the floor system on the stresses in truss members is not considered in common truss analyses, but, since the floor beams and stringers are integral parts of the structure, they will to some extent influence the behavior under load of the bridge as a whole. Floor-system behavior under load can be visualized in the following manner if the floor beams are initially regarded as rigid:

1. Stringers and bottom chords elongate as the bridge deflects.
2. Since the floor beams are actually quite flexible in the horizontal plane, the initial tension imparted to the stringers by the deflection of the bridge will be reduced when the floor beams are relaxed and allowed to deflect laterally. If the end floor beams are first released, the stringer tensile loads in the end panel will be reduced by this relaxation, creating an unbalance in the stringer tension in the panels adjacent to the next floor beam. In this manner, stringer tension in each successive panel is reduced as the floor beams are relaxed in turn until the entire system is balanced. The floor-beam deflection will decrease from the ends to the center of the bridge, where from symmetry the deflection is zero. Correspondingly, the stringer tensile loads will increase in successive panels from ends to the center as the relaxation from lateral deflection of the floor beams diminishes.

The total tensile stress acting in the bottom plane of the truss in any particular panel must include whatever tension remains in the stringers after relaxation. Since the stringers are invariably located near the top flanges of the floor beams, their tensile load acts in a horizontal plane some distance above the centroid of the bottom chord. Thus, the resultant of the combined bottom chord and stringer tensile stresses is also somewhat above the centroid of the bottom chord, in effect decreasing the effective depth of the truss. This decrease in the arm of the internal resisting couple necessitates a compensating increase in each force component of that couple; thus the top chord compressive stress will increase as a result of floor-system participation.

In a pony truss bridge the top chord is not laterally braced and consequently acts as a beam column that will fail by buckling at some critical load less than that required for compression failure when fully restrained against lateral deflection. The increase, therefore, in top-chord stresses caused by floor-system participation may significantly reduce the factor of safety against buckling.

The main objective of this paper is to evaluate the relative magnitude and importance of the increase in top chord stresses caused by floor-system participation. For the experimental work covered in this report, the model pony truss bridge erected for the buckling studies was used. The model was adapted by the addition of continuous longitudinal stringers connected to the top flange of the transverse floor beams. The experimental work consisted of load tests on the model bridge both with and without stringers. In addition, the experimental results were compared with a theoretical analysis developed in an effort to predict the behavior of pony truss bridges with continuous longitudinal stringers.

EXPERIMENTAL WORK

The experimental portion of this project consisted of load testing a model pony truss bridge in which a floor system that included longitudinal stringers was incorporated, and then repeating the test with the same load increments for the model bridge without stringers.

Description of the Model.—The model bridge used in the experimental testing consisted of two Warren trusses ten panels in length connected by double-channel floor beams and simply supported on rollers at the end bearings. Lateral bracing made of round bar stock was provided in the bottom plane of each panel. The equivalent of longitudinal stringers was added by clamping 4 in. by $\frac{1}{2}$ in. rectangular bars to the top flanges of the floor-beam channels. Except for the stringers, all structural members were fabricated from standard rolled structural shapes. High strength bolts torqued to 110-ft lb with a calibrated wrench, together with heavy nuts and case-hardened washers, were used throughout at all connections.

The model bridge was complete in all details except for the omission of a deck slab. The proportions of the model were approximately those of a modern two-lane highway bridge of 200-ft span. To insure proper relative stiffness of the web members, the flanges of verticals and diagonals were reduced in width, and holes were cut in the webs of the diagonals. The layout of the bridge and details of members and joints are shown in Fig. 1. Fig. 2 is a photograph of the model bridge.

The incorporation of the longitudinal stringers into the floor system of the model involved devising a clamp that would connect the stringer bars to the floor beams without allowing slippage as tensile loads developed. Because of lack of sufficient space between the floor-beam channels, a yoke clamping the stringers directly over the floor-beam axis was not feasible. It was necessary instead to stagger the connecting clamps on either side of the axis. Holes were drilled through the top flanges of each floor-beam channel to accommodate the clamp bolts. The position of the clamp with respect to the floor-beam axis alternated from panel to panel so that a "long" span was followed by a "short" span, and, since the clamps were staggered across each floor beam, the sequence for the inside stringers was opposite to that for the outside stringers. The details of the clamp arrangement are shown in Fig. 3.

The size and number of stringers provided were based on an average ratio of stringer area to bottom chord area in several representative pony truss bridges for which plans were available. Each stringer was continuous for the entire length of the bridge and was extended a few inches beyond the flange of

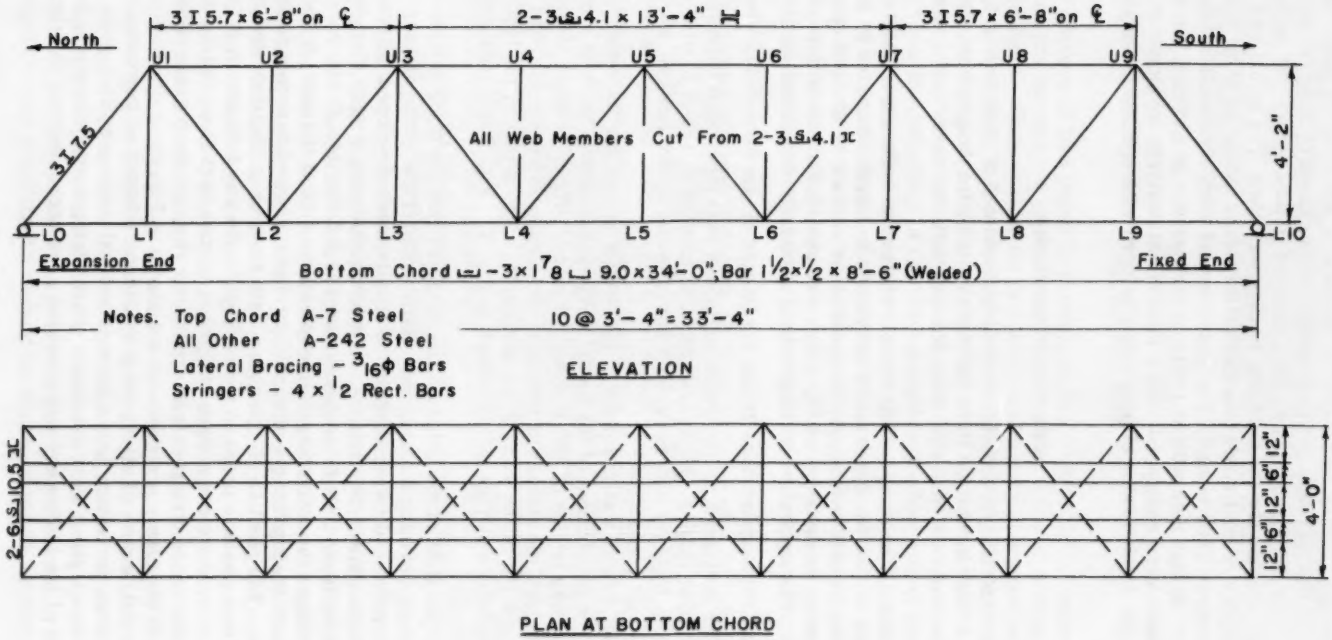

3 I 5.7 x 6'-8" on ℄
2-3 ⧎ 4.1 x 13'-4" ⊥
3 I 5.7 x 6'-8" on ℄
North
South
U1 U2 U3 U4 U5 U6 U7 U8 U9
3 I 7.5
All Web Members Cut From 2-3 ⧎ 4.1 ⊥
4'-2"
L0 L1 L2 L3 L4 L5 L6 L7 L8 L9 L10
Expansion End
Bottom Chord ⊏ - 3 x 1 7/8 ⌊ 9.0 x 34'-0"; Bar 1 1/2 x 1/2 x 8'-6" (Welded)
Fixed End
Notes. Top Chord A-7 Steel
All Other A-242 Steel
Lateral Bracing - 3/16 ϕ Bars
Stringers - 4 x 1/2 Rect. Bars
10 @ 3'-4" = 33'-4"
ELEVATION
2-6 ⧎ 10.5 ⊥
12" 6" 12" 6" 12"
4'-0"
PLAN AT BOTTOM CHORD

FIG. 1.—GENERAL LAYOUT OF PONY TRUSS BRIDGE MODELS

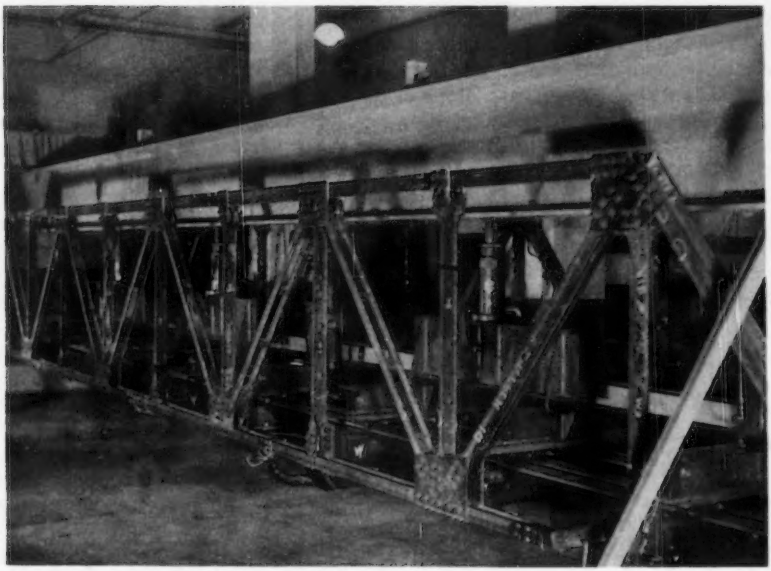


FIG. 2.—OVER-ALL VIEW OF MODEL BRIDGE

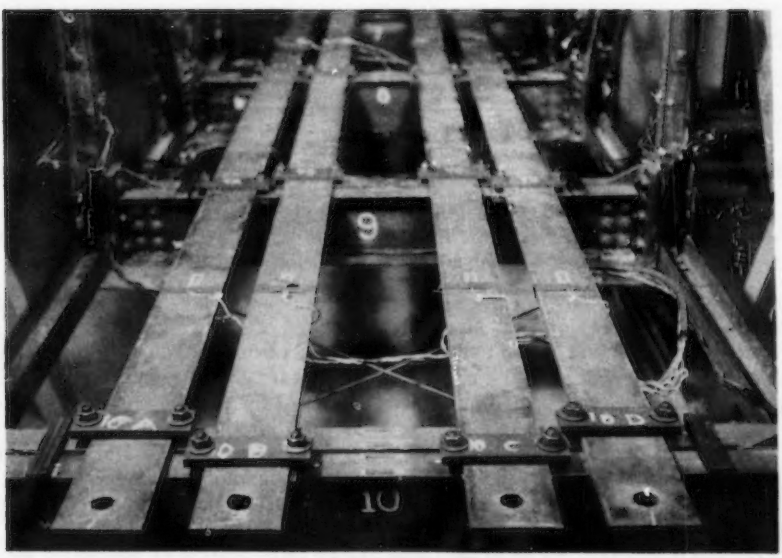


FIG. 3.—STRINGER CONNECTIONS

the end floor beams. A butt weld near the center was required to provide this continuity because the greatest length of bar available was 20 ft. Bending along the longitudinal axis of stringers occurred as the clamps were tightened because of the slight downward slope of the top face of each floor-beam channel flange. However, this bending, which followed a sine wave pattern from panel to panel, was effectively minimized by the insertion of shims beneath the stringers along the edge of the flanges.

Strain Measurements.—Experimental strains in top chords, bottom chords, and stringers were measured with type A-1 SR-4 strain gages by means of a Baldwin type L strain indicator. Each gage was connected to a four-pole switch wired so as to break both sides of the active and the compensating circuits. Two gage stations, each consisting of four symmetrically placed gages, were

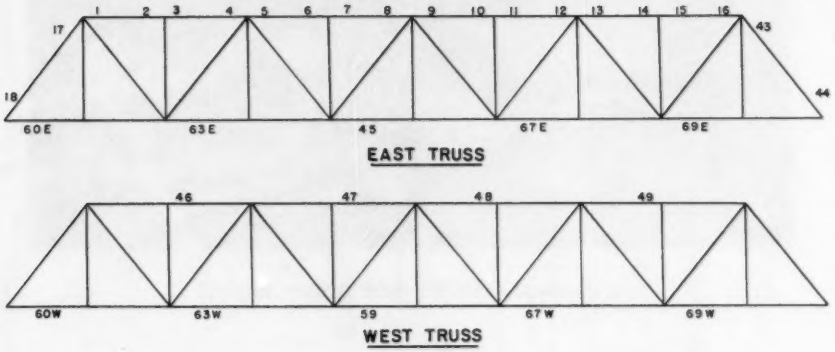


FIG. 4.—STRAIN GAGE STATIONS

established in each panel of the top chord on the east truss. In addition, four similar stations were spaced along the top chord of the west truss. Stations of four gages each were also established on the bottom chords of both the east and west trusses at the center of alternate panels. In the remaining panels a gage was placed on the top and bottom face of each of the four longitudinal stringers, again at the center of the panel. In all, a total of 160 strain gages was employed in obtaining the experimental data. The location of gage stations on the various members is shown in Fig. 4, while the pattern of individual gages at a station for each type of member is shown in Fig. 5. The switch boxes and strain indicators used in taking the strain readings are shown in Fig. 6.

Loading Apparatus.—The bridge was loaded by a system of hydraulic jacks working against a loading beam suspended over the longitudinal axis of the

bridge from a rectangular frame at either end. The bridge roller supports rested on bottom transverse beam connecting the two columns of each end frame; thus, the loading beam and the bridge formed a closed system, and no reaction other than that due to dead weight was transferred to the floor of the testing room.

The downward reaction of the hydraulic jacks was transferred to the floor beams by two intermediate sets of beams. The longitudinal "spanner" beams placed in alternate panels supported the jacks and transferred their loads to the transverse "spreader" beams placed on top of each floor beam. The spreader beam, in turn, divided its load equally between the two trusses, applying it to the floor beam 16 in. on either side of the longitudinal center line of the bridge. Grooved cushion blocks and a roller pin were used at all load transfer points in this system. At panel point six the load was applied directly to the spreader beam by supporting the jack on a pedestal. The position of the jacks on the spanner beams was arranged so that a uniformly distributed load could be applied to the bridge. The arrangement of the jacks and of the supporting system of spanner and spreader beams is shown in Fig. 7.

The load applied to the bridge was controlled by means of a strain gage dynamometer inserted between each jack and the loading beam. These dynamometers were steel cylinders $4\frac{1}{2}$ in. long and $1\frac{1}{2}$ in. in diameter, with four strain gages cemented at equal intervals along the circumference. The four gages were connected in a way that allowed direct reading of the average strain. A covering of beeswax and resin was applied to the cells as protection for the gages. A dynamometer is shown in position on top of a jack in Fig. 7.

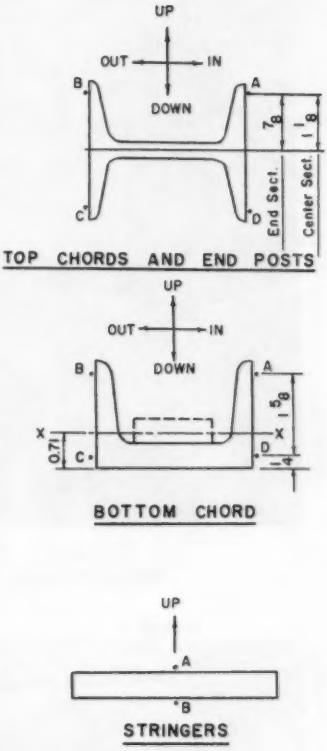


FIG. 5.—STRAIN GAGE LOCATIONS

The dynamometers were calibrated in a universal testing machine to obtain a load-strain relation. A conversion factor of 53.096 lb per micro-in. was obtained from these tests.

Testing Procedure.—After several preliminary tests were conducted to insure satisfactory operation of the strain-measurement equipment, the bridge was tested both with and without stringers under the same loads for each case. Three increments of uniformly distributed load were selected so that a gradual increase in top-chord unit stress to a maximum of about 15,000 psi would result. This upper limit was established as an estimated average value of working stress in top-chord members of pony truss bridges. Heavier loads were not attempted because of the possibility of slippage at the stringer connections as well as the undesirability of approaching the critical top chord

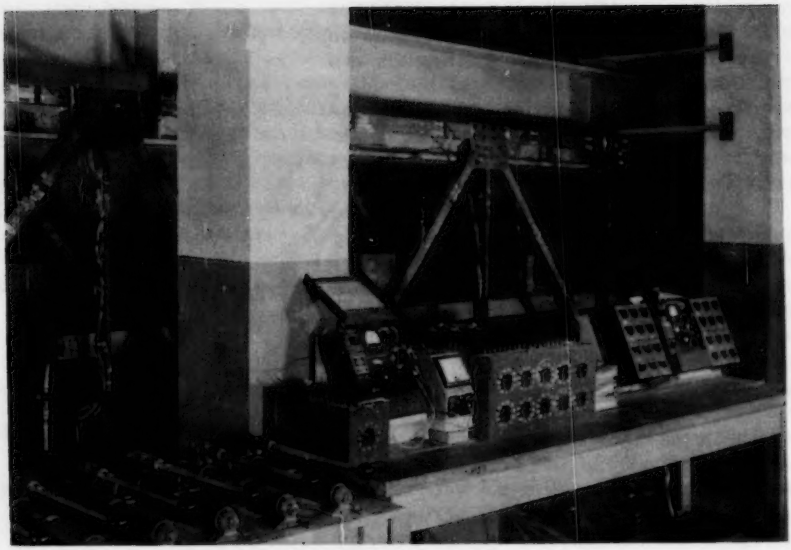


FIG. 6.—STRAIN INDICATORS AND SWITCH BOXES

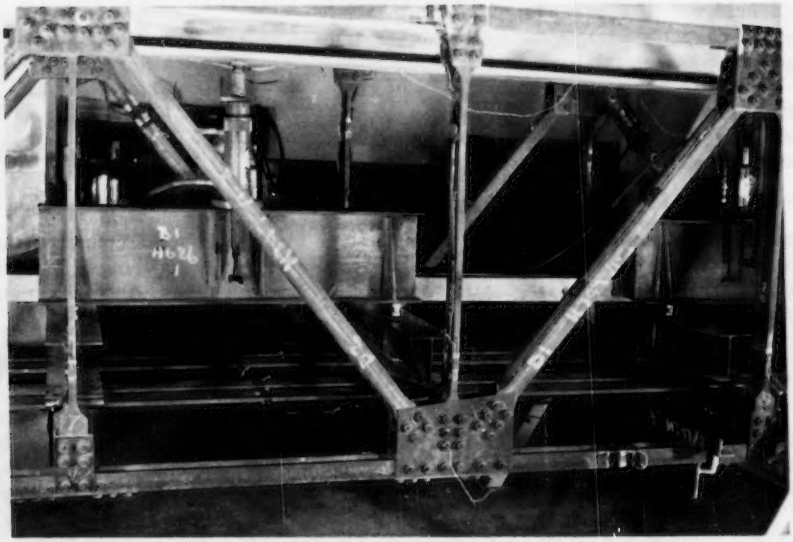
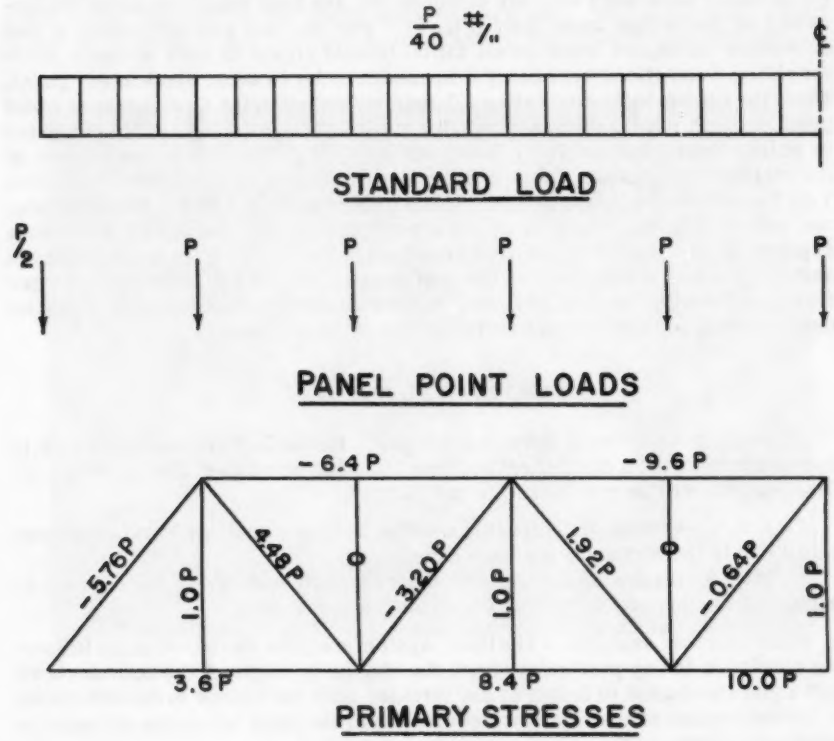


FIG. 7.—LOADING ARRANGEMENT

buckling load. The panel point loads for each increment are shown in Fig. 8. The required dynamometer strain readings for each increment were computed from these panel point loads by use of the conversion factor mentioned previously.

The load increments were applied to the bridge in the same manner for both tests. After zero readings had been taken, the first load increment was applied



LOAD	P(lbs)	f_{tc} (psi)
1	2240	8690
2	2930	11370
3	3630	14085

FIG. 8.—LOAD DIAGRAM

by loading the jacks one at a time until the required dynamometer strain reading for each jack was attained. Several cycles were necessary to apply the desired load over the entire bridge because an increase in load on one jack would somewhat decrease the load on the other jacks. When the dynamometer

strain readings for each jack finally matched the desired reading, all gages were read, and the second load increment was applied in the same manner. After the third load increment had been applied and all gages had been read, the load on the jacks was gradually released until no load remained on the bridge. Then final zero readings for all gages were taken to complete the test.

For the test conducted without stringers, all clamp connection bolts were loosened and removed so that, although still in place on top of the floor beams, the stringer bars were entirely disconnected and thus would not affect the behavior of the bridge under load (Fig. 9). For the test with stringers, it was necessary to impart some small initial tensile strain in each stringer while the clamp connections were being tightened in order to avoid slack in any panel. First the clamp bolts connecting all four stringers to the floor beam at panel point five were securely tightened. Then each stringer was slightly elongated by pulling from one end with a lever inserted through a hole in the portion of the stringer overhanging the end floor beam. The tension required to impart 1 or 2 micro-in. of strain in each stringer as read on the strain indicator was maintained with the levers while the clamp connection bolts on the floor beam adjacent to the central panel were tightened. Then the levers were released and the procedure repeated for the next panel. In this way, when all stringer clamp connections were tightened, an approximately equal amount of initial tensile strain existed throughout the length of each stringer.

THEORETICAL ANALYSIS

To predict the effect of floor-system participation on top-chord stresses in pony truss bridges, a theoretical analysis has been developed. For convenience this analysis will be considered in two parts:

1. A floor-system analysis that results in theoretical loads in the bottom chord and in the stringers for each panel.
2. The calculation of the top-chord stresses in each panel, using the resultant of the theoretical bottom chord and stringer loads.

Floor-System Analysis.—The floor-system analysis was based on the following condition: In any particular panel, the change in length of the bottom chord will equal the change in length of any stringer plus the change in the difference in the deflections of the adjacent floor beams at the point where the stringer in question is connected.

The following basic equation can be written from this condition:

$$\Delta L_{bc} = \Delta L_s + \Delta d_{fb} \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad (1)$$

in which ΔL is the change in length in inches, and Δd represents the difference in deflection of the adjacent floor beams. The following subscripts denote the member to which the symbol pertains: bc denotes the bottom chord, s represents the stringer, and fb denotes the floor beam. In Eq. 1 either the inside or outside stringer can be considered as long as the corresponding value of the differential floor-beam deflection is used.

The floor-beam deflection can be analyzed as the combination of three separate deflections. Each of these components can be expressed as functions of either stringer tensile load or applied panel point load, and a set of simultaneous equations can then be written and solved for the unknown stringer loads.

The components of floor beam deflection are evaluated separately in the following sections.

Lateral Deflection Due to Bending.—The flexural deflection of the floor beams under unbalanced stringer loads in adjacent panels was evaluated by means of the conjugate-beam method. The floor beam was assumed as being hinged at the diaphragm connections on either end. The effective length was taken as 39 in., the distance between the centroids of the bolts connecting the floor beam channels to the diaphragms on each end. The moment of inertia about the vertical axis of the floor beam channels was calculated to be 5.20 in.^{4}.

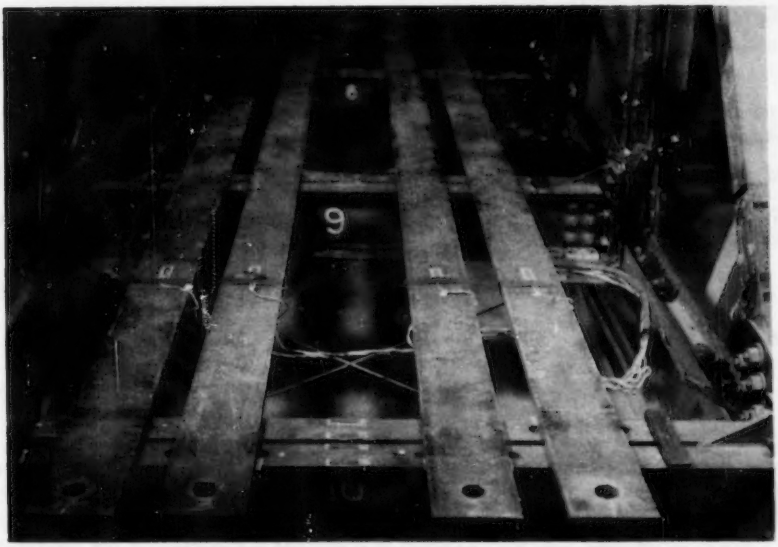


FIG. 9.—STRINGER CLAMPS DISCONNECTED

A modulus of elasticity for the steel of 29,500 ksi was used. The calculation of flexural deflections for the points where the inside and outside stringers are connected resulted in the following expressions:

$$d_o = 5.317\ F_o + 7.957\ F_i \quad \ldots\ldots\ldots\ldots \quad (2a)$$

and

$$d_i = 7.957\ F_o + 12.474\ F_i \quad \ldots\ldots\ldots\ldots \quad (2b)$$

in which F is the force exerted on the floor beam by the difference in stringer tensile load in the adjacent panels, and the subscripts o and i refer to the outside and inside stringers.

Lateral Deflection of the Top Flange Due to Torsional Rotation.—Since the stringers are connected to the top flanges of the floor beam, the tensile stringer loads are eccentric with respect to the floor beam axis and, therefore, create some torsional rotation. The angular rotation θ at a distance L from the end of a member with a torsional rigidity K that is subjected to a torque T can be calculated from:

$$\theta = \frac{T\ L}{G\ K} \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad (3)$$

The shearing modulus G of steel was assumed as 12,000 ksi. The torsional rigidity of the floor-beam channels was calculated using an approximate formula for rolled sections:

$$K = \frac{1.10}{3} \Sigma b\ h^3 \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad (4)$$

A torsional rigidity K of 0.668 in.^{4} was determined, considering the web thickness of the two channels combined. For an effective floor beam length of 39 in. and a stringer load eccentricity of $3\frac{1}{4}$ in. from the axis of twist at the centroidal axis of the floor beam, the following expressions for deflection due to torsional rotation were calculated:

$$d_o = 9.883\ F_o + 9.883\ F_i \quad \ldots\ldots\ldots\ldots\ldots \quad (5a)$$

and

$$d_i = 17.840\ F_o + 30.263\ F_i \quad \ldots\ldots\ldots\ldots\ldots \quad (5b)$$

Lateral Translation of the Top Flange Due to Floor Beam Rotation at Joints.—The change in angle of the bottom chord at each panel point occurring as the bridge deflects under load will tend to rotate the floor beams at each joint. This rotation of the floor beam about an axis approximately intersecting the centroid of the bottom chord will result in a lateral translation of the top flange of the floor beam which can be calculated in terms of the panel point load by the elastic-weight method. The end floor beam will be rotated the same amount as the change in angle of the bottom chord at the end panel point, while the intermediate floor beams will be rotated only one-half the total change in angle of the bottom chord at the panel point. The translation of the top flange can be calculated by multiplying the angular rotation of the floor beam by the distance between the center of twist and the centroid of the stringers, which was estimated as $5\frac{1}{2}$ in. For a uniform panel point load P, the difference in translation between adjacent floor beams was calculated, and the following results were obtained:

Panel	$\Delta\ d_{jr} \times 10^{-6}$
1	+0.2247 P
2	-0.4773 P
3	-1.6602 P
4	-1.6761 P
5	-1.6444 P

where the subscript jr refers to the joint rotation component of the floor-beam deflection.

Each component of floor-beam deflection having been evaluated, these expressions were substituted into the basic equation which is now considered in the following slightly expanded form:

$$\Delta L_{bc} - \Delta d_{jr} = \Delta L_s + \Delta d_f + \Delta d_t \quad \ldots\ldots\ldots\ldots \quad (6)$$

The subscripts f and t refer to the flexural and torsional components of floor-beam deflection respectively. For convenience, the component of floor-beam deflection due to joint rotation, which is a function of panel-point load, was considered as a correction to the left-hand side of Eq. 6. Thus all terms on the right-hand side of Eq. 6 can be expressed directly in terms of stringer tensile load, the elongation of the stringer being defined by:

$$\Delta L_s = \frac{S_s\ L}{A\ E} = 0.678\ S_s \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad (7)$$

in which S is the axial stress, in pounds.

From symmetry, only one quarter of the floor system plan need be considered, and a set of ten simultaneous equations can be written using the basic equation as modified previously. Numbers are inserted as superscripts to denote the panel to which each symbol pertains:

$$\Delta L^1_{bc} - \Delta d^1_{jr} = 0.678\ S^1_o + 15.200(2\ S^1_o - S^2_o) + 17.840(2\ S^1_i - S^2_i) \quad \ldots\ldots \quad (8a)$$

$$\Delta L^1_{bc} - \Delta d^1_{jr} = 0.678\ S^1_i + 25.797(2\ S^1_o - S^2_o) + 42.737(2\ S^1_i - S^2_i) \quad \ldots\ldots \quad (8b)$$

$$\Delta L^2_{bc} - \Delta d^2_{jr} = 0.678\ S^2_o + 15.200(2\ S^2_o - S^1_o - S^3_o) + 17.840(2\ S^2_i - S^1_i - S^3_i) \ldots \quad (8c)$$

$$\Delta L^2_{bc} - \Delta d^2_{jr} = 0.678\ S^2_i + 25.797(2\ S^2_o - S^1_o - S^3_o) + 42.737(2\ S^2_i - S^1_i - S^3_i) \ldots \quad (8d)$$

$$\Delta L^3_{bc} - \Delta d^3_{jr} = 0.678\ S^3_o + 15.200(2\ S^3_o - S^2_o - S^4_o) + 17.840(2\ S^3_i - S^2_i - S^4_i) \ldots \quad (8e)$$

$$\Delta L^3_{bc} - \Delta d^3_{jr} = 0.678\ S^3_i + 25.797(2\ S^3_o - S^2_o - S^4_o) + 42.737(2\ S^3_i - S^2_i - S^4_i) \ldots \quad (8f)$$

$$\Delta L^4_{bc} - \Delta d^4_{jr} = 0.678\ S^4_o + 15.200(2\ S^4_o - S^3_o - S^5_o) + 17.840(2\ S^4_i - S^3_i - S^5_i) \ldots \quad (8g)$$

$$\Delta L^4_{bc} - \Delta d^4_{jr} = 0.678\ S^4_i + 25.797(2\ S^4_o - S^3_o - S^5_o) + 42.737(2\ S^4_i - S^3_i - S^5_i) \ldots \quad (8h)$$

$$\Delta L^5_{bc} - \Delta d^5_{jr} = 0.678\ S^5_o + 15.200(S^5_o - S^4_o) + 17.840(S^5_i - S^4_i) \quad \ldots\ldots\ldots \quad (8i)$$

$$\Delta L^5_{bc} - \Delta d^5_{jr} = 0.678\ S^5_i + 25.797(S^5_o - S^4_o) + 42.737(S^5_i - S^4_i) \quad \ldots\ldots\ldots \quad (8j)$$

In Eqs. 8 the coefficients evaluating floor-beam deflection for the flexural and torsional cases have been combined, and the unbalanced force F has been

expressed directly as the difference in stringer tensile loads in the panels adjacent to the floor beams at both ends of the panel under consideration. The final form of the equations after expanding and arranging terms is:

$$\Delta L_{bc}^{5} - \Delta d_{jr}^{5} = 15.878\, S_o^5 + 17.840\, S_i^5 - 15.200\, S_o^4 - 17.840\, S_i^4 \quad \text{(9a)}$$

$$\Delta L_{bc}^{5} - \Delta d_{jr}^{5} = 25.797\, S_o^5 + 43.415\, S_i^5 - 25.797\, S_o^4 - 42.737\, S_i^4 \quad \text{(9b)}$$

$$\Delta L_{bc}^{4} - \Delta d_{jr}^{4} = -15.200\, S_o^5 - 17.840\, S_i^5 + 31.078\, S_o^4 + 35.680\, S_i^4 - 15.200\, S_o^3 - 17.840\, S_i^3 \quad \text{(9c)}$$

$$\Delta L_{bc}^{4} - \Delta d_{jr}^{4} = -25.797\, S_o^5 - 42.737\, S_i^5 + 51.594\, S_o^4 + 86.152\, S_i^4 - 25.797\, S_o^3 - 42.737\, S_i^3 \quad \text{(9d)}$$

$$\Delta L_{bc}^{3} - \Delta d_{jr}^{3} = -15.200\, S_o^4 - 17.840\, S_i^4 + 31.078\, S_o^3 + 35.680\, S_i^3 - 15.200\, S_o^2 - 17.840\, S_i^2 \quad \text{(9e)}$$

$$\Delta L_{bc}^{3} - \Delta d_{jr}^{3} = -25.797\, S_o^4 - 42.737\, S_i^4 + 51.594\, S_o^3 + 86.152\, S_i^3 - 25.797\, S_o^2 - 42.737\, S_i^2 \quad \text{(9f)}$$

$$\Delta L_{bc}^{2} - \Delta d_{jr}^{2} = -15.200\, S_o^3 - 17.840\, S_i^3 + 31.078\, S_o^2 + 35.680\, S_i^2 - 15.200\, S_o^1 - 17.840\, S_i^1 \quad \text{(9g)}$$

$$\Delta L_{bc}^{2} - \Delta d_{jr}^{2} = -25.797\, S_o^3 - 42.737\, S_i^3 + 51.594\, S_o^2 + 86.152\, S_i^2 - 25.797\, S_o^1 - 42.737\, S_i^1 \quad \text{(9h)}$$

$$\Delta L_{bc}^{1} - \Delta d_{jr}^{1} = -15.200\, S_o^2 - 17.840\, S_i^2 + 31.078\, S_o^1 + 35.680\, S_i^1 \quad \text{(9i)}$$

$$\Delta L_{bc}^{1} - \Delta d_{jr}^{1} = -25.797\, S_o^2 - 42.737\, S_i^2 + 51.594\, S_o^1 + 86.152\, S_i^1 \quad \text{(9j)}$$

For a given panel-point load and appropriate values for the elongation of the bottom chord in each panel, Eqs. 9 can be solved for the ten unknown stringer tensile loads. Cholesky's method was adopted for the simultaneous solution because the tabular setup facilitated machine computations. Furthermore, this method was extremely advantageous for multiple solutions, since only a few additional steps were required to solve for different values inserted in the left-hand side of the equation, once the conversion matrix for the constant coefficients of the right-hand side had been calculated.

Several cycles were required to obtain the theoretical bottom chord and stringer tensile loads for each load increment used in the experimental tests. Initially the bottom chord elongation calculated from a primary-stress analysis for the given load was inserted in the left-hand side of the equations, and a first approximation of the final theoretical stringer tensile loads was obtained by simultaneous solution. In the next cycle the initial bottom chord elongation was reduced by an amount equivalent to a loss in bottom-chord load equal to the combined inside and outside stringer load calculated in the first cycle. In each succeeding cycle, the bottom-chord stress was adjusted by the change in combined stringer load between the two preceding cycles. The results of the fifth cycle were taken as the final theoretical stringer tensile loads since the correction at that point became negligible. The final theoretical bottom chord loads were computed from the values of bottom-chord elongation used in the solution of the fifth cycle. In each cycle, the bottom chord elongations were corrected for floor-beam deflection due to joint rotation calculated for the given panel-point load with the previously derived expressions.

Calculation of Theoretical Top Chord Stresses.—Having evaluated the combined stringer loads in each panel by means of the floor-system analysis, the

theoretical top-chord stress can be determined in a simple manner. The increase in top-chord stress due to floor-system participation can be computed by adding to the moment about any bottom-chord joint an increment of moment due to the eccentricity of the unbalanced stringer loads at each joint. Thus, at each bottom chord joint, an external moment equal to the combined stringer load in the adjacent panel times an arm of 5.54 in., the distance between bottom chord and stringer centroids, is applied. The top-chord stress in each panel can then be calculated by taking moments about the appropriate bottom-chord joint as in the common method of sections, resulting in:

$$S_{tc} = \frac{M + \Delta M}{h} \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad (10)$$

in which M is the moment due to the applied panel-point loads, ΔM represents the increment of moment due to the unbalanced stringer loads = $5.54\ S_s$, and h is the depth of truss, 50 in.

Calculation of Theoretical Top-Chord Stresses by an Approximate Method.— The theoretical top-chord stresses can be calculated with virtually the same results by an entirely different approach, without using an elaborate floor-system analysis. This analysis is based on the assumption that the resultant of the bottom chord and stringer loads acts at the centroid of the bottom-chord and stringer areas.

Using this assumption, the decrease in the effective depth of the truss Δh can be found by:

$$\Delta h = \frac{5.54\ A_s}{A_s + A_{bc}} \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad (11)$$

in which A denotes the cross-sectional areas in square inches and 5.54 is the distance, in inches, from the centroid of the stringers to the centroid of the bottom chord. The top-chord stresses S_{tc} can now be closely approximated by taking moments about this centroid at the appropriate bottom chord joints:

$$S_{tc} = \frac{M}{h - \Delta h} \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad (12)$$

in which M is again the moment due to the applied loads. Since Δh is usually very small in comparison to the total depth of the truss h, the moment due to the eccentricity of the diagonal loads about the moment center can be neglected without appreciably affecting the results.

This approach to the problem, which in effect assumes equal unit stresses in the bottom chord and stringers, results in a greater decrease in the effective depth of truss and a corresponding increase in the predicted top chord stress.

TEST RESULTS

The experimental axial loads in the top chord, bottom chord, and stringers were determined for each load increment from the strain-gage readings in order to compare the results of the tests with and without stringers and to check the predictions of the theoretical analysis. The first step in the reduction of the experimental data was the calculation of unit strains for each gage, using an average of the zero readings taken before and after loading. Then the axial strain at the centroid of the section at each gage station was computed from the individual strains for each gage at the particular station.

For the top chord and stringer stations this axial strain was a simple average of the strains for the individual gages, which were equidistant from the centroid of the section. For the bottom-chord stations it was necessary to weight the individual gage strains in proportion to their distance from the horizontal centroidal plane to obtain the axial strains at the centroid. Next, the axial strains in all members for a typical quarter of the bridge were computed by averaging the axial-strain values for all gage stations in corresponding locations, as to panel and member, throughout the bridge. Finally the experimental axial loads in each member of this typical portion of the bridge were readily calculated by:

$$S = e A E \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad (13)$$

in which e is the average axial unit strain. A modulus of elasticity E of 29,500 ksi was used for all members. Handbook values were used for the cross-sectional area of the bottom chord and stringers, but average values computed from measurements with micrometer and calipers were used for the top-chord sections.

Precision.—The precision in strain measurement varied from station to station in proportion to the magnitude of the strains. The difference in zero readings taken before and after loading seldom exceeded 12 micro-in. but averaged about 6 micro-in. for all gages. Reference gages under no load showed a corresponding shift during the loading period and thus indicated a uniform change, so an average of both zero readings was used in the calculation of unit strains. The error introduced by this shift was estimated to range from less than 1% to a maximum of about 2% of unit strain for the bottom chord and top chord stations but was substantially higher for the stringer stations, particularly in the end panels, where strains were rather small. The uncertainty of this portion of the data was not considered serious, however, because of the relatively close agreement of the stringer strains in each of the several preliminary load tests.

For the top chord and bottom chord stations, each composed of four symmetrically placed gages, it was possible to check the consistency of the data against the Navier hypothesis that plane sections remain plane after bending. The average error for this condition at each station was calculated as one-quarter of the sum, taken with alternating signs, of the four measured strains at the particular station. The average error computed in this manner for the top and bottom chord stations was ordinarily less than 1% of the axial strain. On the whole, then, strain measurement in the experimental testing was carried out with an average error of about 2% of axial strain, which is the range of precision to be expected with the equipment used.

Experimental Results.—The experimental top-chord stresses for each increment of both the test with stringers and the test without stringers are presented in Table 1, which also includes the experimental end-post stresses entered in the line for the first panel. Since the average applied panel-point load for corresponding increments of the two tests did not agree exactly, the experimental stresses for the test without stringers were adjusted by a factor equal to the ratio of the average applied load for the test with stringers to the average applied load for the test without stringers, in order to eliminate the effect of small differences in loading conditions between the two tests. The experimental stresses for each increment are also plotted in Fig. 10, which

shows that the change in stress in each panel from initial to final load increment is linear, as would be expected.

The increase in top-chord stress resulting from floor system participation is shown in Table 1 as a percentage based on the experimental values for the test without stringers. This percentage increase varies from an average of about 5% for the first load increment to an average of about 2% for the final load increment. Thus the experimental results indicate that the increase in top-chord stress due to floor-system participation is small for stresses within the ordinary working range and that the increase may be even less significant for higher stresses approaching the critical buckling load.

Comparison of Experimental and Theoretical Results.—The experimental stringer and bottom chord loads, together with the theoretical combined stringer

TABLE 1.—EXPERIMENTAL TOP-CHORD STRESSES IN POUNDS

Ld (1)	Panel (2)	S_{TC} without Stringers (3)	S_{TC} with Stringers (4)	Increase (5)	% Increase (6)
1	1	12,210	12,900	690	5.65
	2	13,800	14,470	670	4.85
	3	13,800	14,420	620	4.50
	4	22,070	23,210	1,140	5.17
	5	21,840	22,990	1,150	5.27
2	1	15,970	16,420	450	2.82
	2	18,020	18,440	420	2.33
	3	17,970	18,440	470	2.62
	4	28,740	29,485	745	2.59
	5	28,450	29,340	890	3.13
3	1	19,940	20,070	130	0.65
	2	22,390	22,660	270	1.20
	3	22,290	22,660	370	1.66
	4	35,510	36,270	760	2.14
	5	35,230	35,980	750	2.13

loads for each increment of the test with stringers, are presented in Table 2. The theoretical and experimental values of combined stringer tensile load in each panel for the final load increment are plotted in Fig. 11a, and the points are connected by smooth curves to provide a convenient comparison. Although the ratio of outside to inside stringer loads calculated by the theoretical floor system analysis is considerably higher than the experimental ratio, the combined values correspond rather well. The theoretical combined loads follow a nearly parabolic curve from panel to panel, with the greatest discrepancy from the experimental values occurring in the end panel where the error in strain readings was relatively high. Considering the number of assumptions required in the evaluation of the floor-beam-deflection components, the agreement between the theoretical and experimental combined stringer loads is remarkably good. Since the combined stringer load, not the individual inside and outside stringer loads, is the significant value in the theoretical determination of the effect of floor-system participation on the top-chord stresses, the results of the floor-system analysis should furnish a satisfactory basis for the remaining portion of the analysis.

The theoretical top-chord stresses for each increment of the test with stringers are presented in Table 3. In Col. 3, top-chord stresses calculated by a primary stress analysis are shown, while in Col. 4 the top-chord stresses calculated by the complete theoretical analysis that accounted for floor-system participation are shown. Comparison of these values indicates the relatively

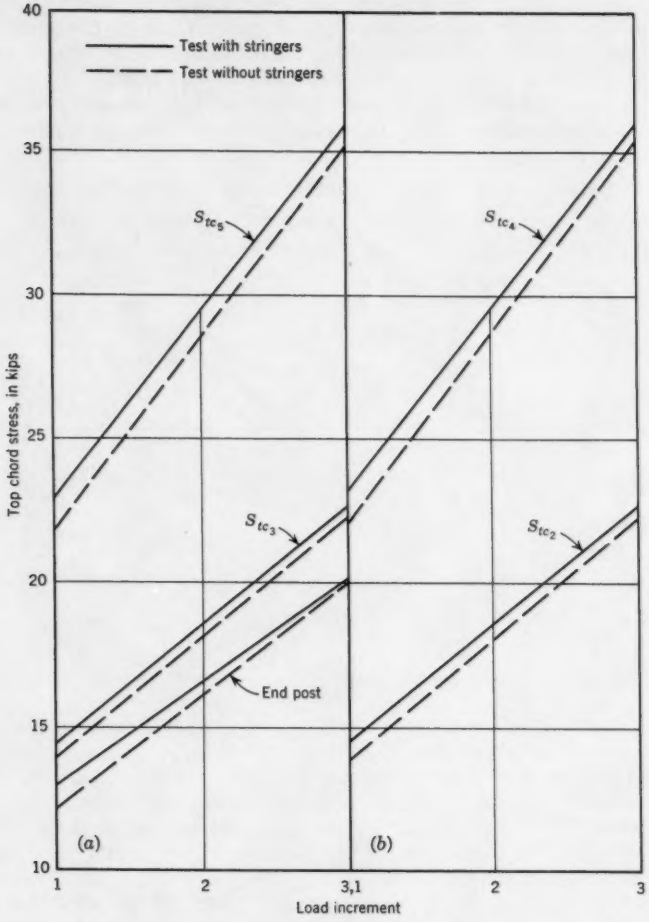


FIG. 10.—EXPERIMENTAL TOP-CHORD STRESSES

small amount of increase in top-chord stress contributed by the unbalanced stringer loads. Comparison of the primary values with the corresponding experimental values for the test without stringers, listed in Col. 3 of Table 1, shows slightly erratic but on the whole good agreement, as would be expected. The percentage deviation of the theoretical values for the complete analysis

from the corresponding experimental values for the test with stringers is also shown in Table 3. The percentage deviation varies from extremes of +4.3% to -6.1%, an over-all range of about 10%. The average deviation in the end panels is +2.85%, while the average in the center panels is -3.60%.

The theoretical analysis was supplemented by a crude, but simple, approximate analysis, described above, which yielded higher predicted values of top-chord stress. These approximate values are listed in Col. 6 of Table 3, along with the percentage deviation from the experimental, which averaged +5.0% for all panels. The approximate analysis was based on the assumed resultant of the bottom chord and stringer areas. The fact that the approximate values, in spite of the crudeness of the basic assumption, compare favorably in accuracy

TABLE 2.—EXPERIMENTAL AND THEORETICAL STRINGER STRESSES IN POUNDS

Ld	Panel	Experimental S_o	Experimental S_i	Experimental S_s	Theoretical S_s	Experimental S_{bc}	Experimental $S_{bc} + S_s$	Primary S_{bc}
(1)	(2)	(3)	(4)	(5)	(6)	(7)	(8)	(9)
	1	472	118	590	924	7,320	7,910	8,050
	2	1,357	826	2,183	1,605	6,620	8,803	8,050
1	3	1,416	885	2,301	2,205	16,510	18,811	18,760
	4	1,770	1,357	3,127	2,574	15,890	19,017	18,760
	5	2,006	1,298	3,304	2,742	19,100	22,400	22,390
	1	472	59	531	1,211	9,190	9,721	10,540
	2	1,357	826	2,183	2,104	8,180	10,363	10,540
2	3	1,770	1,121	2,891	2,890	20,640	23,530	24,585
	4	2,124	1,475	3,599	3,372	20,330	23,730	24,585
	5	2,478	1,416	3,894	3,592	24,500	28,390	29,280
	1	649	236	885	1,514	11,210	12,095	13,080
	2	1,770	1,003	2,773	2,632	10,360	13,130	13,080
3	3	2,242	1,298	3,540	3,616	25,470	29,010	30,530
	4	2,537	1,711	4,248	4,221	25,080	29,330	30,530
	5	2,950	1,888	4,838	4,496	30,200	35,040	36,360

with the results of the complete analysis indicates the feasibility of such methods in computing the effect of floor-system participation.

The theoretical values predicted by both the complete analysis and the approximate method, together with the experimental values, are plotted for convenient comparison in Fig. 11 for the final load increment.

COMPARISON WITH OTHER TESTS

Field tests have been conducted on pony truss bridges where measured top-chord stresses have been compared with theoretical top-chord stresses computed from a primary analysis. A notable example of these field tests are the tests conducted on the Fort Loudon, Pa., bridge. Results from the Fort Loudon tests yield top-chord stresses in member U4 U5 24.9% higher than the computed stress that was based on a primary analysis.

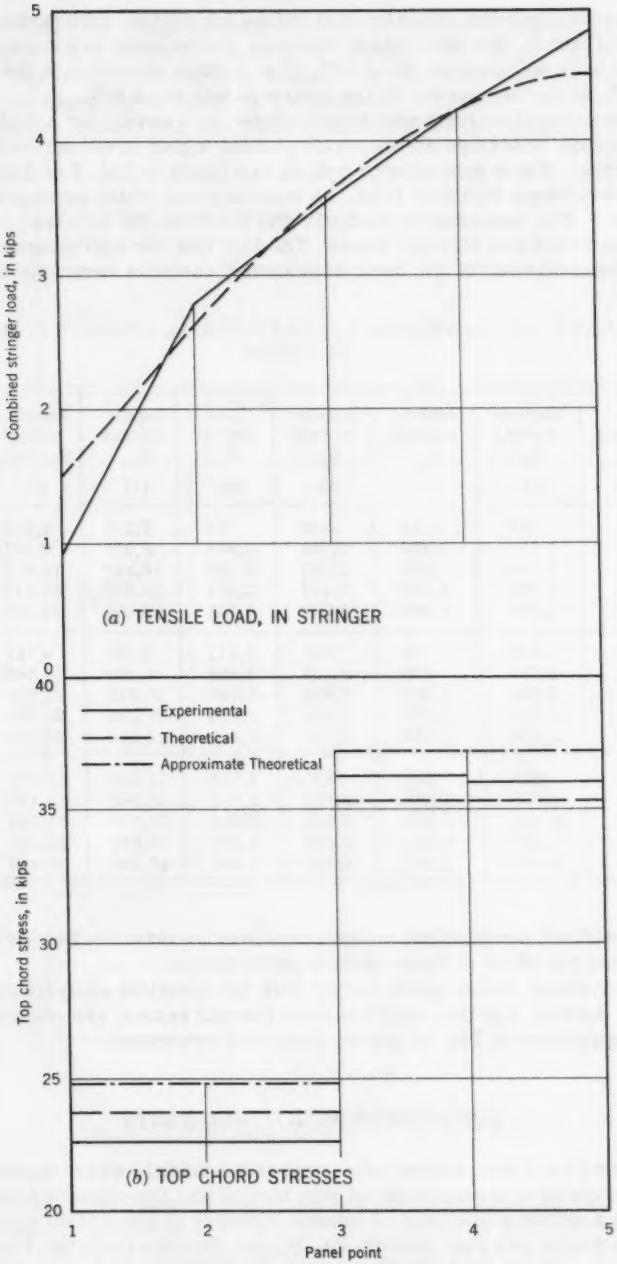


FIG. 11.—EXPERIMENTAL AND THEORETICAL RESULTS FOR THIRD LOAD INCREMENT

The Fort Loudon bridge was complete in all details including stringers, a continuous concrete floor slab and lateral bracing in the plane of the bottom chords. The model bridge did not contain a complete floor system, the floor slab was omitted, and the lateral bracing consisted of 1/4 in. round, cold-rolled steel bars. These differences in construction probably account for a large part of the discrepancy between the test results on the two bridges.

The approximate analysis developed in this paper was applied to the Fort Loudon bridge. For the loading condition considered, the results of the approximate analysis yielded top chord stresses of 2,350 psi, considering the stringers and bottom chords as effective area, and 2,500 psi, considering the stringers, floor slab, and bottom chords as effective area. These results were compared with a theoretical primary stress of 2,290 psi, and an average measured stress of 2,860 psi. When the complete floor system, consisting of stringers and floor slab, was considered in the approximate analysis, it was

TABLE 3.—THEORETICAL TOP-CHORD STRESSES

Ld	Panel	S_{tc}, Statical	S_{tc} with Stringers	% Deviation	S_{tc} with Stringers, Approximate Method	% Deviation
(1)	(2)	(3)	(4)	(5)	(6)	(7)
1	2	14,285	14,463	0	15,348	+ 6.1
	3	14,285	14,529	+0.8	15,348	+ 6.4
	4	21,485	21,770	-6.2	23,025	- 0.8
	5	21,485	21,789	-5.2	22,855	- 0.6
2	2	18,725	18,958	+2.8	20,090	+ 9.0
	3	18,725	19,045	+3.3	20,090	+ 9.0
	4	28,105	28,479	-3.4	30,135	+ 2.2
	5	28,105	28,503	-2.9	29,915	+ 2.0
3	2	23,255	23,543	+3.9	24,915	+10.0
	3	23,255	23,651	+4.3	24,915	+10.0
	4	34,910	35,344	-2.6	37,375	+ 3.0
	5	34,910	35,375	-1.7	37,105	+ 3.1

found that the top-chord stress increased by 9.2%, which is considerably lower than the 24.9% increase which was found by field test measurements. Results from the laboratory model test indicate that the percentage increase in top-chord stresses varies inversely with the applied loads. If the Fort Loudon bridge had been tested under heavier loads it is conceivable that the percentage increase in top-chord stress might have been lower than the 24.9% which was found.

Differences in testing techniques between the Fort Loudon bridge and the model bridge should also be considered when comparing test results. The Fort Loudon tests were conducted on a full-scale bridge subjected to moving loads, and test results were compared with computed values to obtain the percentage increase in top-chord stress. Tests on the model bridge were conducted in a laboratory on a model bridge which was capable of being tested with the stringers connected and with the stringers disconnected, and the percentage increase in top-chord stress was computed by comparing results of tests performed

under both conditions. In the tests conducted on the model with the stringers connected, a very small initial tension was applied to the stringers to insure stringer participation at all increments of loading. By measuring this initial tension before and after loading, it was found that slippage in the stringer clamps did not occur under load.

CONCLUSIONS

Since the model bridge did not contain a floor system that was complete in all details, the following conclusions can be applied only to the model tested:

1. Experimental tests on the model bridge indicated an average increase in top-chord stresses resulting from floor-system participation of 2% to 3% in the ordinary range of working stresses.
2. The experimental results indicated that the increase in top-chord stress resulting from floor-system participation varies inversely with the applied loads. These stress increases varied from an average of 5% for low loads to an average of 2% to 3% for loads which produced stresses in the ordinary working range.
3. The increase in top-chord stresses due to floor-system participation may be closely approximated by using for the effective center line of the bottom chord the centroid of the areas of the bottom chords and the longitudinal floor members combined.

RECOMMENDATIONS

Since the tests on the model bridge verify the assumption that floor-system participation causes an increase in top-chord stresses, further tests should be conducted to determine the effect of a floor slab and lateral bracing. Floor-system participation will affect the top-chord and bottom-chord stresses in through trusses as well as pony trusses, so any future testing should not be restricted to pony truss bridges. The effect of the lateral bracing in the plane of the top chords of through trusses should also be considered in future tests.

ACKNOWLEDGMENTS

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The writers are grateful for the helpful guidance of Committee G of the Column Research Council.

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DYNAMIC EFFECTS OF EARTHQUAKES[a]

By Ray W. Clough,[1] F. ASCE

SYNOPSIS

The principal factors controlling the dynamic response of structures to earthquakes are summarized, and are related to the lateral force provisions recommended for inclusion in the Uniform Building Code by the Structural Engineers Association of California (SEAOC). These provisions are seen to conform very well with the concepts of dynamic theory.

INTRODUCTION

Experience with recent earthquakes in Tehachapi, Calif. and Mexico City, Mexico has shown that it is possible to build economical, attractive structures which are highly resistant to earthquake affects. But at the same time, these earthquakes demonstrated that where the dynamic effects of earthquakes are not fully understood or properly accounted for, the results can be disastrous.

The purpose of this paper is to summarize the principal factors controlling the dynamic response of structures to earthquakes, and to relate these factors to current trends in the development of the earthquake provisions in building codes. It will be seen that the lateral force requirements recently recommended by the SEAOC take cognizance of the major factors affecting the dynamic response of structures and, thus, provide a rational basis for the design of earthquake-resistant structures.

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[a] Presented at February, 1959 Convention in Los Angeles, Calif.

[1] Assoc. Prof. of Civ. Engrg., Univ. of Calif., Berkeley, Calif.

Before going into the details of the dynamic response theory, it will be useful to emphasize a few pertinent facts regarding the nature of earthquakes. An earthquake is, of course, simply a ground-vibration phenomenon. Since the earth is elastic in its gross characteristics, and possesses mass, it will vibrate when subjected to a shock loading just as will any other mechanical system. Thus, when a slippage occurs suddenly at a fault zone, shock waves are propagated through the earth in all directions, and when the surface manifestations of these waves pass any given point on the earth, it (and any structure located on it) will be caused to vibrate. Motions induced by the ground vibrations may have both vertical and horizontal components, but since buildings normally have considerable excess strength in the vertical direction it is customary to consider only the effects of the horizontal motions in earthquake-resistant design. It should be emphasized that the forces developed during an earthquake

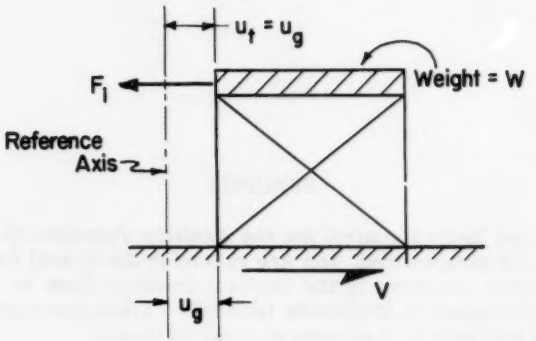


FIG. 1.—EARTHQUAKE FORCES, RIDGID STRUCTURE

are not applied directly to the structure, but rather are inertia forces resulting from the motions of the structure.

EARTHQUAKE EFFECTS ON A RIGID STRUCTURE

In order to provide a suitable background for this study of the dynamic-response problem, it will be useful to consider first the effect of an earthquake on a rigid structure. Such a structure is shown in Fig. 1. It is assumed that both the building and its foundations are rigid so that the earthquake motions of the ground, u_g, are transmitted directly to the building. In this case, it is clear that an effective earthquake force, F_i, will be developed in the structure equal to the product of the ground acceleration and the mass of the structure,

$$F_i = \ddot{u}_g \frac{W}{g} \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad (1)$$

in which $\ddot{u}_g$ is the ground acceleration, W denotes the weight of the structure, and g is the acceleration of gravity. For convenience, Eq. 1 is usually rearranged so that the force is given as the product of the weight of the structure

and a seismic coefficient, C, which represents the ratio of the ground acceleration to the acceleration of gravity:

$$F_i = \frac{\ddot{u}_g}{g} W = C W \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad (2)$$

in which

$$C = \frac{\ddot{u}_g}{g} \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad (3)$$

For design purposes, it is common practice to express the earthquake force in terms of the shearing force developed at the base of the structure. In this case, simple statics show that the base shear, V, is equal to the force F_i, and is given by

$$V = C W \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad (4)$$

Eq. 4 demonstrates that the dynamic analysis of a rigid structure is very simple. All that is required is an estimate of the maximum ground acceleration which will occur during the earthquake. This acceleration, expressed as a ratio to the acceleration of gravity is the seismic coefficient C in the formula.

The rigid-structure concept provided the basis for the lateral-force provisions of some of the earliest earthquake codes, which specified that a structure should be designed for a certain percentage of gravity (say 10% or 12%), regardless of the characteristics of the structure. Unfortunately, the dynamic-response characteristics of actual structures are not so simple. Their flexibility and mass impart to them vibration characteristics which directly affect the magnitude of the seismic forces to which they will be subjected during an earthquake.

DYNAMIC RESPONSE OF A FLEXIBLE STRUCTURE

The effect of a structure's flexibility on its response may be discussed most easily by reference to a simple, one-story structure, as shown in Fig. 2. The weight of the structure, W, is assumed to be concentrated at the roof level. Such a structure is said to have a single degree of freedom (considering plane motion only) because only one type of deformation is possible, represented here by the displacement, u. The significant dynamic properties of this structure, in addition to its weight, are the stiffness of the columns, k, which represents the force developed per unit displacement, and the damping, c, which represents the force per unit velocity. In the explanation which follows, damping will be omitted for simplicity, but the effect of damping will be included with the final results.

In the absence of damping, the base shear in this structure may be expressed as the product of the displacement and the column stiffness,

$$V = k u \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad (5)$$

Dynamic equilibrium conditions (using d'Alembert's principle) show that the base shear must balance the inertia force of the mass, that is,

$$\frac{W}{g}\ddot{u}_t + k\,u = 0 \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad (6)$$

(noting the sign convention assumed in Fig. 2). It will be noted that the inertia force here depends on the total motion of the mass, rather than the ground motion, as was the case only in Fig. 1. It is convenient to express the total acceleration as the sum of the ground acceleration and the relative acceleration of the mass with respect to the ground, thus

$$\ddot{u}_t = \ddot{u}_g + \ddot{u} \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad (7)$$

Then Eq. 6 may be rewritten as

$$\frac{W}{g}\ddot{u} + k\,u = -\frac{W}{g}\ddot{u}_g = F_e \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad (8)$$

Eq. 8 is identical with that which would apply to a stationary structure subjected to an effective force, F_e, equal to the product of the mass of the structure and the ground acceleration. Thus the dynamic effects of earthquakes

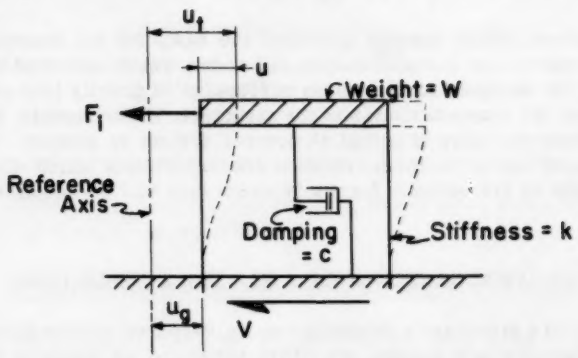


FIG. 2.—EARTHQUAKES FORCES, FLEXIBLE STRUCTURE

may be studied by considering the structure to be stationary and applying to it an effective earthquake force, F_e.

Now it is clear that this effective force is not directly resisted by shears in the columns; the mass must first be accelerated, and, thus, the inertia of the structure modified the dynamic effect of the applied load. The base shear, V, in this case, depends on the nature of the applied force F_e (that is, the time history of the ground acceleration) and also on the vibration characteristics of the structure. If the ground displacements were a simple harmonic motion of period T_p, as shown in Fig. 3, the effective earthquake force could be expressed

$$F_e = F_o \sin p\,t = -\ddot{u}_{go}\frac{W}{g}\sin p\,t \quad \ldots\ldots\ldots\ldots \quad (9)$$

in which

$$p = \frac{2\pi}{T_p} \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad (10)$$

In this case, the maximum (steady state) base-shear force is given by

$$V_{max} = u_{max}\, k = F_o \left[\frac{1}{1 - \left(\frac{T}{T_p}\right)^2}\right] = W \frac{\ddot{u}_{go}}{g} \left[\frac{1}{1 - \left(\frac{T}{T_p}\right)^2}\right] \quad \ldots (11)$$

in which

$$T = 2\pi \sqrt{\frac{W}{g\,k}} \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad (12)$$

is the period of vibration of the structure. From Eq. 11 it is clear that the response of the structure depends, in a very direct fashion, on the natural period

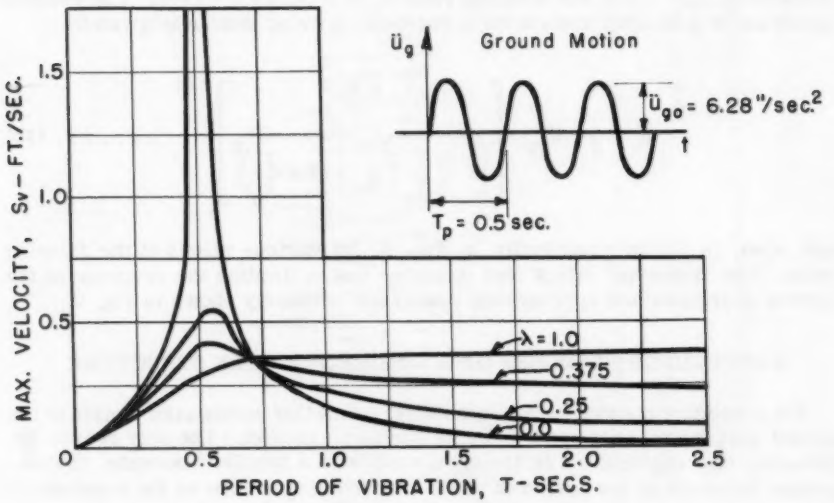


FIG. 3.—RESPONSE TO HARMONIC MOTION

of vibration of the structure, which depends, in turn, on its stiffness and weight. The base shear may be either less than or greater than that of a rigid structure, depending on how close to resonance this system is, that is, how near its natural period of vibration is to the period of the applied ground motion.

It is sometimes convenient to express the response of a structure to a specific ground motion in terms of a velocity coefficient, S_v, as follows:

$$V_{max} = \frac{W}{g} \frac{2\pi}{T} S_v \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad (13)$$

where S_v represents the maximum velocity produced in the structure by the particular motion. The velocity coefficient for simple harmonic ground motion, therefore, is given by

$$S_v = \frac{T}{2\pi} \ddot{u}_{go} \left[\frac{1}{1 - \left(\frac{T}{T_p}\right)^2} \right] \quad \ldots\ldots\ldots\ldots\ldots (14)$$

A graph representing the variation of the velocity coefficient, S_v, with the period of vibration of the structure for a harmonic ground motion of a particular amplitude and period, is shown in Fig. 3. Such a graph is called the velocity spectrum of the ground motion because it shows the maximum velocity developed by this motion for a complete spectrum of periods of vibration of the structure.

It will be recalled that the preceding remarks refer to the response of an undamped structure. If the structure is damped, that is, if it has some form of resistance which depends on the velocity of motion, as represented by the viscous damper shown in Fig. 2, the magnitude of this damping force will also affect the response of the structure. The magnitude of the viscous damping of a system, c, is usually expressed as a ratio to a critical or reference damping coefficient, c_c. Thus, the damping ratio λ, is given by $\lambda = c/c_c$. The velocity spectrum of a damped system for a harmonic-ground motion is given by

$$S_v = \frac{T}{2\pi} \ddot{u}_{go} \left[\frac{1 + \left(2\lambda \frac{T}{T_p}\right)^2}{\left[1 - \left(\frac{T}{T_p}\right)^2\right]^2 + \left(2\lambda \frac{T}{T_p}\right)^2} \right]^{1/2} \quad \ldots\ldots\ldots (15)$$

and, also, is shown graphically in Fig. 3, for various values of the damping ratio. The important effect that damping has in limiting the response of the system at frequencies approaching resonance is clearly shown in Fig. 3.

EARTHQUAKE RESPONSE OF A SIMPLE FLEXIBLE STRUCTURE

The preceding material is not intended to imply that earthquake motion of the ground may be represented by simple harmonic motion. The only reason for including this explanation is to emphasize, with a familiar example, the important influence of the period of vibration of the structure on its response to a given ground motion. That an earthquake is far from a simple harmonic motion is clearly shown by Fig. 4, which presents the ground acceleration measured at Taft, Calif., from the Tehachapi earthquake of July 21, 1952. The motion may be characterized best as a series of erratic, almost random, acceleration pulses. Thus, the concept of resonance which was applied to harmonic motions has no place in the treatment of earthquake response.

On the other hand, the response to an earthquake motion can be expressed in terms of a velocity spectrum, just as was described previously, if the velocity spectrum is determined properly; and, again, the response will be found to depend on the period of vibration of the structure. For a completely arbitrary ground motion, such as is represented by an earthquake, the velocity

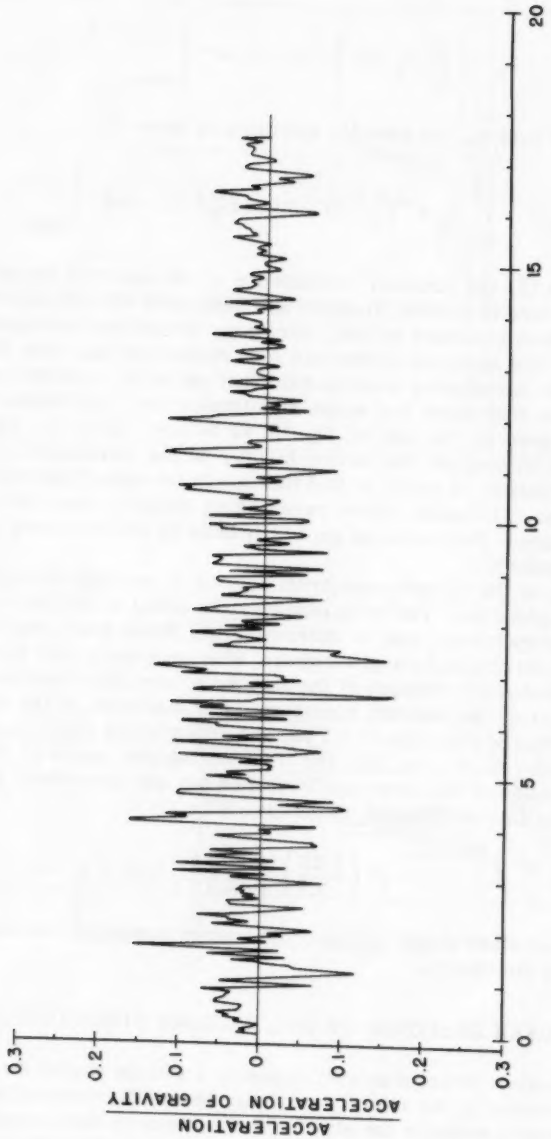


FIG. 4.—ACCELEROGRAM RECORDED AT TAFT, CALIFORNIA JULY 21, 1952 (COMPONENT N21E)

spectrum of an undamped system may be evaluated from

$$S_v = \left[\int_0^t \ddot{u}_g \sin \frac{2\pi}{T} (t - \tau)\, d\tau \right]_{max} \quad \ldots\ldots\ldots (16)$$

while for a damped system, the velocity spectrum is given by

$$S_v = \left[\int_0^t \ddot{u}_g\, e^{-\lambda\left(\frac{2\pi}{T}\right)(t-\tau)} \sin \frac{2\pi}{T} (t - \tau) d\tau \right]_{max} \quad \ldots\ldots (17)$$

In both Eqs. 16 and 17, the primary dependence of the spectral values on the period of the structure is evident. Damped and undamped velocity spectra calculated for the motion recorded at Taft, during the Tehachapi earthquake, are shown[2] in Fig. 5. The spectral curves are less regular in this case than they were for the simple harmonic-ground motion because of the erratic nature of the earthquake, but they have the same significance and the maximum base shear can be obtained by the use of Eq. 13, as before. Since Eq. 16 (or 17) must be evaluated throughout the entire history of the earthquake, for any given period of vibration, in order to find the maximum velocity developed for that one period, the calculation effort required to obtain a complete velocity spectrum is enormous. Such work is generally done by either analog or automatic digital computers.

The importance of the velocity-spectrum concept in earthquake engineering cannot be over-emphasized. The complete dynamic effect of the earthquake is represented by the spectrum, and to determine the force which would be developed in a given structure, by a given quake, it is necessary only to evaluate the damping and period of vibration of the structure, and then find the appropriate value of S_v from the velocity spectrum. For example, if the structure of Fig. 2 had a period of vibration of 0.7 sec and 10% critical damping, the corresponding spectral velocity, S_v, for the Taft earthquake would be 0.9 ft per sec. Then, if the mass of the structure were 20 kips, the base shear produced in this structure by this earthquake motion would be

$$V = \frac{W}{g} \frac{2\pi}{T} (0.9) = 20 \left(\frac{6.28}{0.7} \right) \left(\frac{1}{32.2} \right) (0.9) = 5.0 \text{ kips}$$

The response of any other single-degree-of-freedom system to this earthquake could be evaluated similarly.

EARTHQUAKE RESPONSE OF MULTI-STORY STRUCTURES

Although the dynamic effect of an earthquake on a simple elastic structure is completely represented by the velocity spectrum, there still remains the important question of how to evaluate the effect of earthquakes on more complex systems such as multi-story buildings. Fortunately, the procedure developed for structures having a single degree of freedom may be applied similarly to multiple-degree-of-freedom systems. It is necessary only to evaluate first the vibration properties of the structure, that is, its vibration periods and mode

[2] "Behaviour of Structures During Earthquakes," by G. W. Housner, Proceedings, ASCE, Vol. 85, No. EM 4, Ocotober, 1959, p. 109.

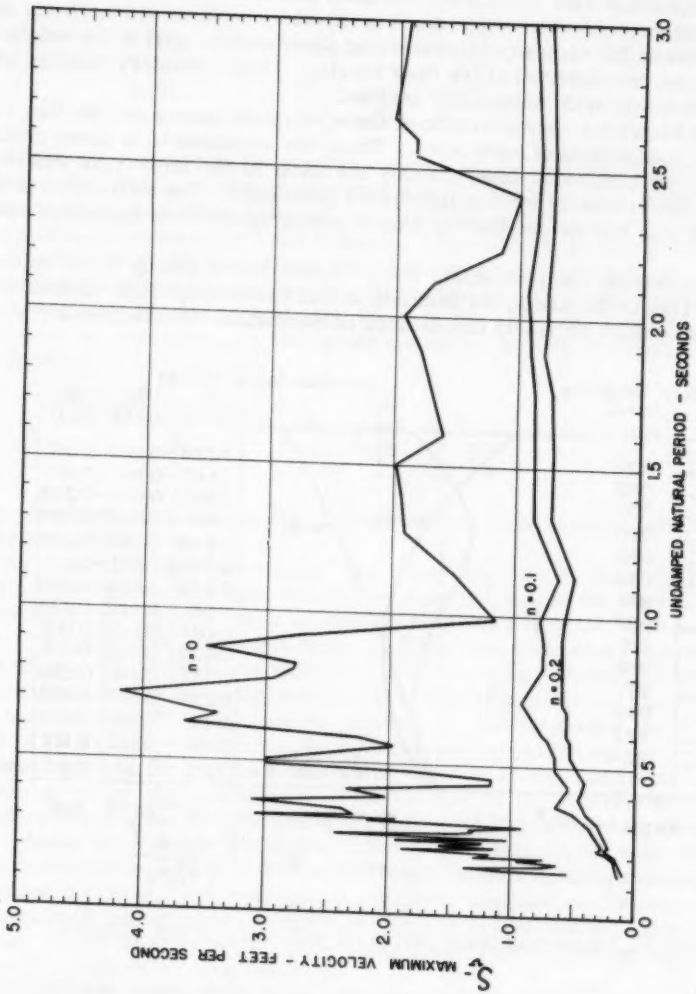


FIG. 5.—VELOCITY SPECTRUM FOR TAFT EARTHQUAKE, JULY 21, 1952

shapes. A structure may vibrate with as many different mode shapes and periods as it has degrees of freedom, and a multi-story building will have one degree of freedom for each story (considering plane motion only) if the weight is assumed to be concentrated at the floor levels. Thus a 10-story building will have ten vibration mode shapes and periods.

Now the important characteristic of these vibration modes is that they are completely independent of each other. Thus, the response to a given ground motion can be calculated independently for each mode, exactly as was described for the one-story system illustrated previously. The total effect of the earthquake may then be obtained by simply adding together the individual mode effects.

To demonstrate the procedure, the structure shown in Fig. 6 will be considered. This is the Alexander Building in San Francisco, Calif. undoubtedly the subject of more technical discussions on earthquake effects than any other

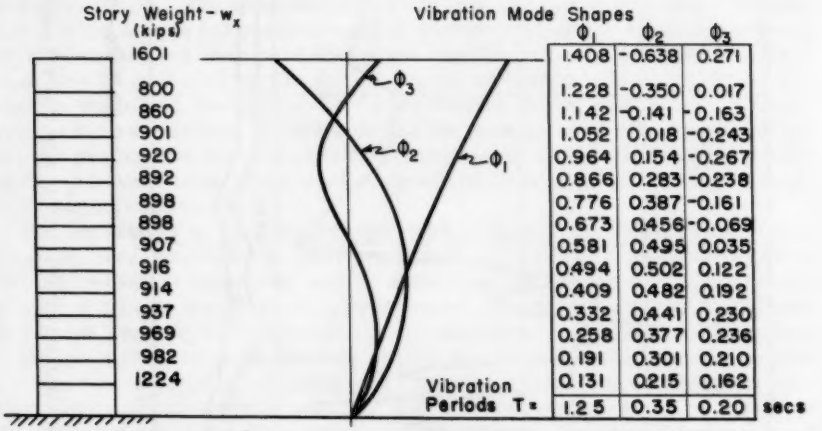


FIG. 6.—VIBRATION PROPERTIES OF THE ALEXANDER BUILDING

building in the world. The first, second, and third mode shapes for this building, and the corresponding vibration periods are shown. Also indicated is the effective weight, W_n, associated with each mode. This weight is used, together with period of vibration and spectral velocity value for each mode, to calculate the base shear for that mode, using the equation

$$V_n = \frac{W_n}{g} \frac{2\pi}{T_n} S_{v_n} \quad \ldots\ldots\ldots\ldots\ldots\ldots (18)$$

which is the same as Eq. 13 except that the subscript n has been added to indicate that values appropriate to the n^{th} mode of vibration are to be used. The

effective-weight for the n^{th} mode is obtained from the relationship

$$W_n = \frac{\left(\sum_x \phi_{xn} w_x\right)^2}{\sum \phi_{xn}^2 w_x} \quad \ldots\ldots\ldots\ldots\ldots\ldots (19)$$

in which ϕ_{xn} represents the displacement at the x^{th} floor level in the n^{th} mode of vibration, and w_x represents the weight of the x^{th} floor. The base shear calculated for each of the three modes of vibration, using the velocity spectrum of the Taft earthquake record and assuming 10% damping, is shown in Fig. 7.

In addition to the base shear for the multi-story structure, of course, the manner in which the forces are distributed through the height of the structure is also required. In general, the force in the n^{th} mode at height x, F_{xn}, is given by the base shear for that mode multiplied by a distribution coefficient, as follows:

$$F_{xn} = V_n \left[\frac{\phi_{xn} w_x}{\sum_x \phi_{xn} w_x} \right] \quad \ldots\ldots\ldots\ldots\ldots\ldots (20)$$

The distribution of forces for the three modes considered in this analysis is also shown in Fig. 6.

As was mentioned previously, the total response of the structure to the earthquake motion may be obtained by superposition of the responses calculated for each mode. Thus, if we had the time-history of the base-shear variation for each mode, the time-history of the total base shear could be determined by merely adding the individual response terms at each instant of time. However, it should be recognized that the total maximum base shear developed by the Taft earthquake cannot be obtained by merely adding the base shears shown in Fig. 6, even though each value represents the maximum force developed in that particular mode. This is because the maximum velocities represented by the velocity-spectrum values for the different periods of vibration would occur at different times during the history of the quake, and thus they do not represent simultaneous affects. Accordingly, the value obtained by direct superposition of the maximum modal forces will always exceed the true maximum forces. For example, a complete analysis of the response of the Alexander Building to the El Centro earthquake of 1940, showed[3] that the superposed modal maxima gave a base shear force which exceeded the true maximum base shear by about 28%.

Since it is possible to obtain only an approximation to the maximum response by direct superposition of the modal maxima, it is equally rational and considerably simpler to calculate only the fundamental mode response, and to increase it by a factor to account for higher mode effects. Referring again to the analysis of the response of the Alexander Building to the El Centro quake, it was found that the true maximum base shear was about 19% greater than that given by the first mode spectral-response value. This increase applies only to this particular building and earthquake, of course, but it may be considered representative of the order of magnitude of higher mode effects in tall buildings. (However, it may be noted here that increasing the fundamental mode

[3] "On the Importance of Higher Modes of Vibration on the Earthquake Response of a Tall Building," by R. W. Clough, Bulletin, Seismological Soc. of Amer., Vol. 45, No. 4, October, 1955, p. 289.

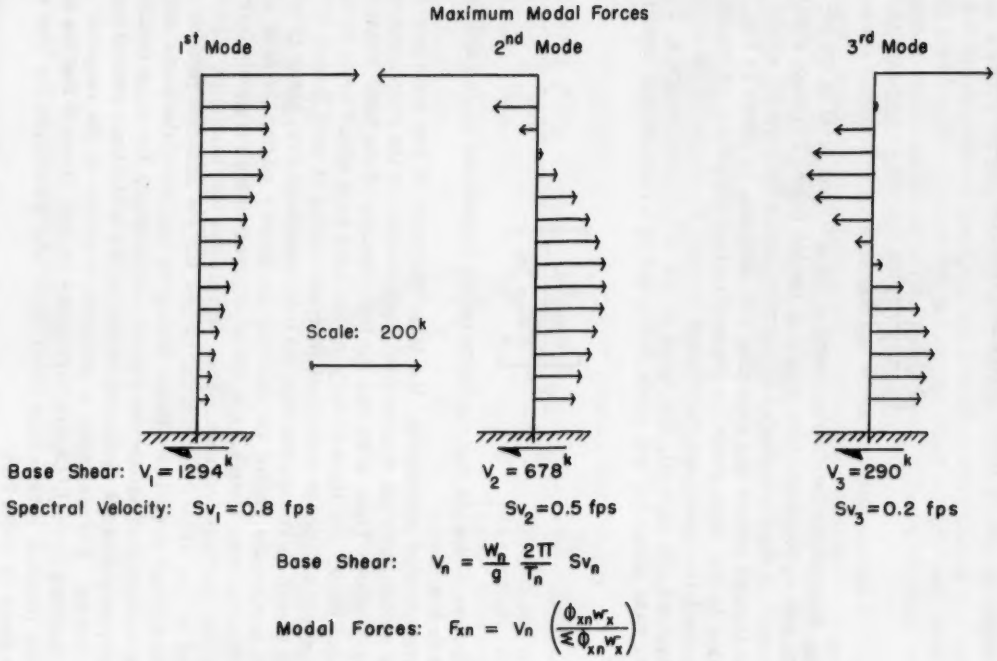

Maximum Modal Forces
1st Mode
2nd Mode
3rd Mode
Scale: 200k
Base Shear: V1 = 1294k
V2 = 678k
V3 = 290k
Spectral Velocity: Sv1 = 0.8 fps
Sv2 = 0.5 fps
Sv3 = 0.2 fps
Base Shear: $V_n = \frac{W_n}{g} \frac{2\pi}{T_n} Sv_n$
Modal Forces: $F_{xn} = V_n \left(\frac{\phi_{xn} w_x}{\sum \phi_{xn} w_x} \right)$

FIG. 7.—RESPONSE OF ALEXANDER BLDG. TO TAFT QUAKE

response by a constant factor to account for higher mode effects is not entirely rational, because the higher modes provide different effects at different heights. A better procedure would be to add a specified fraction, say 50%, of the second and third mode maxima to the first mode maximum to obtain an estimate of the total forces developed throughout the height.)

INELASTIC RESPONSE TO EARTHQUAKES

The procedure described previously makes possible the analysis of earthquake forces in any type of structure, and would apparently provide a complete picture of the dynamic effects of earthquakes. However, when the forces due to a moderately severe quake are calculated by this procedure, it is found that they exceed, by a significant amount, the design forces which would be specified by building codes. For example, in the previously mentioned study of the response of the Alexander Building to the El Centro earthquake of 1940, it was found that the base shear was about 20% of the weight of the building. Even the relatively moderate Taft earthquake would have produced a base shear of about 10% of the buildings' weight. On the other hand, building codes would specify a value of about 3% to 5% for the base-shear coefficient for this structure. This would appear to indicate that the lateral-force provisions of building codes are quite unconservative in providing resistance to a severe quake.

At the same time, however, it must be recognized that buildings having considerably less strength than is required by modern codes have withstood rather severe quakes with only moderate damage. This apparent discrepancy may be attributed, in part, to the fact that buildings possess considerable strength in excess of the design values due to use of conservative design stresses and to the participation of non-structural elements in resisting lateral deformations. Nevertheless, this factor does not fully explain the relatively slight damage exhibited by many ordinary buildings which have gone through heavy quakes. Even more important in many cases, is the fact that, as the response of the building builds up, cracking and yielding begin to take place, and these inelastic deformations absorb a large part of the vibrational energy of the structure. As a result, the continued build-up of energy which is required to develop the maximum velocities indicated by the spectral response curves is prevented.

On this basis, it is clear that inelastic deformations of the structure are a predominate factor in limiting the forces developed in a structure by a strong earthquake. Moreover, it is evident that earthquake codes have empirically taken account of this effect, since the code provisions provide strengths which are not sufficient to resist the earthquake forces elastically. To account for this effect rationally requires that inelastic action be incorporated into the analysis, and this may be accomplished effectively only through the use of automatic digital or analog computers. A study of this type was performed[4] by J. Penzien in which he evaluated the inelastic response of a single-story system to the El Centro earthquake of 1940. A part of the results of this study is presented in Fig. 8, in a form somewhat similar to the velocity-spectrum curves discussed previously (except that maximum displacement rather than velocity is the quantity presented.) The dashed curve in Fig. 8 represents the

4 "Elasto-Plastic Response of a Single Mass System Subjected to a Strong-Motion Earthquake," by J. Penzien, presented at a meeting of the ASCE, Los Angeles, Calif., February, 1959.

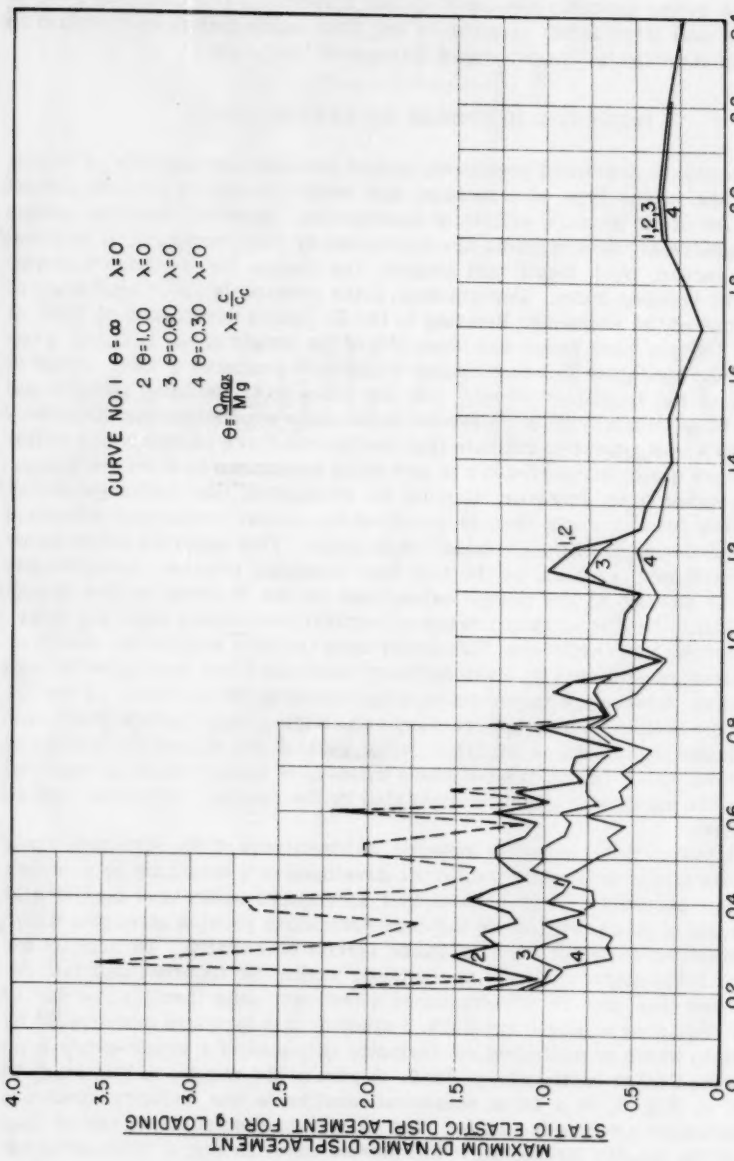
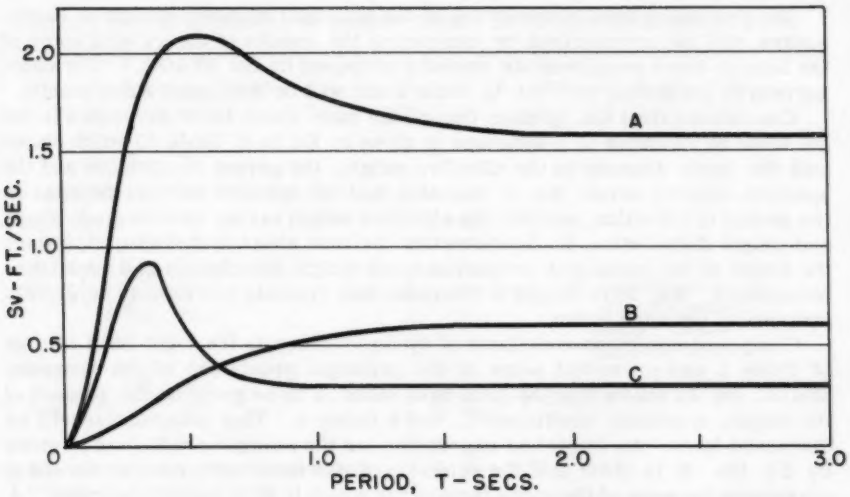

CURVE NO. 1 θ = ∞ λ = 0
2 θ = 1.00 λ = 0
3 θ = 0.60 λ = 0
4 θ = 0.30 λ = 0
θ = ωmax/Mg λ = c/cc
MAXIMUM DYNAMIC DISPLACEMENT / STATIC ELASTIC DISPLACEMENT FOR 1g LOADING
PERIOD (T) - SECONDS
0 0.5 1.0 1.5 2.0 2.5 3.0 3.5 4.0
0 0.2 0.4 0.6 0.8 1.0 1.2 1.4 1.6 1.8 2.0 2.2 2.4
1,2
3
4
1,2,3
4

FIG. 8.—MAXIMUM DYNAMIC RESPONSE FOR EL CENTRO, CALIFORNIA EARTHQUAKE OF MAY 18, 1940 E-W COMPONENT (FROM REFERENCE 3)

maximum elastic response of the structure for varying values of period of vibration, while the solid lines indicate the inelastic response for several values of plastic limit. This limit is represented by the parameter θ, which is the ratio of the lateral force which would initiate yielding to the weight of the structure. Thus, decreasing values of θ indicate decreasing elastic strength. The important effect that inelastic deformations have in limiting displacements is evident in this figure, since the weaker structures are seen to undergo smaller displacements.

DYNAMIC CHARACTERISTICS OF THE EARTHQUAKE

One other aspect of the earthquake-response problem should be considered herein—the dynamic characteristics of the earthquake itself. As was noted



A. ± 25 Miles from center of large earthquake
B. ± 70 Miles from center of large earthquake
C. ± 10 Miles from center of small earthquake

FIG. 9.—UNDAMPED VELOCITY SPECTRUM CURVES (FROM REFERENCE 1)

previously, the velocity spectrum depends on the nature of the earthquake motion and may be quite different for different quakes. Thus, the difficulties of establishing a standard spectrum for use in earthquake codes is obvious. However, enough earthquake records have now been obtained to establish certain general characteristics of the velocity spectra. These average characteristics were evaluated by G. W. Housner and are presented[2] in Fig. 9. Two basic points may be recognized from the curves of Fig. 9. First, comparison of curves A and B shows that propagation of the quake through the ground for a long distance not only reduces the general intensity of the motion (as might be

expected), but of equal importance, it tends to filter out the short-period components of the motion more effectively than the long-period components. Thus, while a nearby earthquake will tend to cause the most severe damage to stiff, short-period structures, a quake at a greater distance will not affect such structures appreciably and will concentrate its effects instead on the flexible, long-period buildings.

The second basic principle is demonstrated by curve C, which shows that small, nearby quakes still further emphasize the short-period components of the motion, and thus may be expected even more exclusively to limit their damaging effects to the short-period, stiff structures. Flexible, tall buildings will show very slight effects from such local quakes.

SUMMARY

The preceding brief presentation of the principal dynamic effects of earthquakes will be summarized by comparing the results of theory with some of the lateral-force requirements recently proposed by the SEAOC.[5] The comparison is presented, in brief, in Table 1 and will be discussed subsequently.

Considering first the dynamic theory, the base-shear force developed in the n^{th} mode of vibration of a structure is given by Eq. Ia of Table 1, which shows that the force depends on the effective weight, the period of vibration and the spectral-velocity value. Eq. Ib indicates that the spectral velocity depends on the period of vibration, and that the effective weight varies with the mode shape and weight distribution. Eq. Ic shows that the base shear is distributed through the height of the building in proportion to the weight distribution and modal displacements. Eq. Id is simply a reminder that dynamic forces may be greatly reduced by inelastic action.

Compared with these basic facts of dynamic theory, in the right-hand column of Table 1 are presented some of the principal provisions of the proposed SEAOC. Eq. IIa shows that the total base shear is to be given by the product of the weight, a seismic coefficient C, and a factor k. This latter factor will be discussed later. An empirical expression for the seismic coefficient is given By Eq. IIb. It is clear that the selection of this coefficient must be carefully considered because of the many factors for which it is intended to account. A primary factor, of course, is the quantity $\frac{2\pi}{T} S_v$ (frequency times spectral velocity) and the variation of this quantity is represented in Eq. IIb by the negative cube root of the period. This, however, is only a part of the task assigned to the expression of Eq. IIb. Other factors which must be represented by the equation are the difference between the effective first mode weight, W_1 and the total weight and the influence of higher modes of vibration (because only a single mode is considered in Eq. Ia). Finally, but still of great importance, the factor C, in Eq. IIa, must take account of the energy-absorption effects of inelastic action which greatly alter the maximum response values. Accounting for all of these factors places a heavy burden on the expression of Eq. IIb, but it would appear to do the job as well as any possible choice on the basis of current knowledge.

Eq. IIc provides for distribution of the calculated base shear through the height of the building. Comparison with Eq. Ic shows that the two are identical

5 "Recommended Lateral Force Requirements," Seismology Committee, SEAOC, presented at a meeting of the SEAOC, Yosemite, Calif., October 2, 1958.

if the vibration displacements increase linearly with height. It will be noted in Fig. 6 that the first mode for the Alexander Building is essentially of this shape. Other buildings may tend to emphasize either the shear or the flexural distortion to a greater extent but this appears to be a reasonable assumption for a typical building of tall, slender proportions.

Finally as shown in Eq. IId, the factor k is assigned a value between limits of 2/3 and 4/3. The purpose of this factor is to account for the varying plastic-deformation capacities of different types of construction. It is evident that considerable amounts of energy must be absorbed in plastic deformations if the response of a structure is to be reduced materially below the amplitude of motion which would be developed elastically. Consequently, it is important that the structure possess adequate capacity for plastic deformation. The "k" value of 2/3 is intended to be applied to structural types which may undergo

TABLE 1.—COMPARISON OF DYNAMIC THEORY WITH PROPOSED SEAOC CODE

I.—Dynamic Theory	II.—Proposed SEAOC Code
a. $V_n = \frac{W_n}{g} \frac{2\pi}{T} S_v$	a. $V = k\ C\ W$
b. $S_v = f(T)$ $W_n = \left(\frac{\Sigma \phi_{xn}\ w_x}{\Sigma \phi_{xn}^2\ w_x} \right)^2$	b. $C = \frac{0.05}{3\sqrt{T}}$ accounts for: velocity spectrum effective weight higher modes inelastic action
c. $F_{xn} = V_n \frac{\Sigma \phi_{xn}\ w_x}{\Sigma \phi_{xn}\ w_x}$	c. $F_x = V \frac{w_x\ h_x}{w_x\ h_x}$
d. Inelastic action greatly affects magnitudes of dynamic forces	d. $k = 0.67$ to 1.33 accounts for varying yield capacity of different types of construction

sizeable amounts of plastic deformation without suffering major damage, while the factor of 4/3 would be applied to structures which can undergo only minor amounts of plastic deformation.

It is apparent from the preceding remarks that current concepts regarding the lateral-force provisions of building codes have advanced considerably beyond the original rigid-structure treatment. The seismology committee which promulgated the lateral-force provisions proposed by the SEAOC has done a remarkable job of relating the practical requirements of a building code to the essential features of dynamic theory. There is still a need for extensive research on the inelastic response of structures, and further studies of the characteristics of earthquakes will be needed to define a standard earthquake. However, it is encouraging to find a proposed building code which so nearly represents the current state of the art in this rapidly developing field.

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STRUCTURAL DIVISION
Proceedings of the American Society of Civil Engineers

DISCONTINUITY STRESSES IN BEAMS ON ELASTIC FOUNDATIONS

by Joseph Penzien,[1] A. M. ASCE

SYNOPSIS

A method of analysis is presented for obtaining discontinuity stresses in beams on elastic foundation and cylindrical shells subjected to axisymmetrical loading. Also presented are the solutions of numerous basic cases which provide the necessary numerical constants required in the proposed method of analysis. Several examples are included.

INTRODUCTION

The deflection of a beam on an elastic foundation and the radial displacement of a longitudinal strip along a cylinder subjected to axisymmetrical loading are given by[2]

$$E\,I\,\frac{d^4w}{dx^4} + k_f\,w = p(x) \quad \ldots\ldots\ldots\ldots\ldots\ldots (1a)$$

Note.—Discussion open until September 1, 1960. To extend the closing date one month, a written request must be filed with the Executive Secretary, ASCE. This paper is part of the copyrighted Journal of the Structural Division, Proceedings of the American Society of Civil Engineers, Vol. 86, No. ST 4, April, 1960.

[1] Assoc. Prof. of Civ. Engrg., Univ. of Calif., Berkeley, Calif.

[2] "Strength of Materials," by S. Timoshenko, Part II, D. Van Nostrand Co., New York, 1956.

and

$$D \frac{d^4w}{dx^4} + \frac{E\,h}{r^2} w = p(x) \quad \ldots\ldots\ldots\ldots\ldots\ldots (1b)$$

Using the notation

$$\beta \text{ (beam)} = \sqrt[4]{\frac{k_f}{4\,E\,I}} \quad \ldots\ldots\ldots\ldots\ldots (2a)$$

and

$$\beta \text{ (cylinder)} = \sqrt[4]{\frac{3(1 - \mu^2)}{r^2\,h^2}} \quad \ldots\ldots\ldots\ldots (2b)$$

the solution of Eqs. 1 for any segment of beam or cylinder having a constant section is given by

$$w(x) = e^{\beta x}(A \cos \beta x + B \sin \beta x) + e^{-\beta x}(F \cos \beta x + G \sin \beta x) + w_p(x) \quad \ldots\ldots\ldots\ldots (3)$$

where $w_p(x)$ is the particular solution to the non-homogeneous equation which results from the loading $p(x)$. By successive differentiation of Eq. 3, the slope, moment, and shear relationships for the case of a beam are expressed as

$$\theta(x) = \beta\, e^{\beta x}[(B + A) \cos \beta x + (B - A) \sin \beta x] + \beta\, e^{-\beta x}[(G - F) \cos \beta x + (-G - F) \sin \beta x] + w_p'(x) \quad \ldots (4)$$

$$M(x) = E\,I\,\beta^2\, e^{\beta x}[2\,B \cos \beta x - 2\,A \sin \beta x] + E\,I\,\beta^2\, e^{-\beta x}[-2\,G \cos \beta x + 2\,F \sin \beta x] + E\,I\,w_p''(x) \quad \ldots (5)$$

and

$$V(x) = E\,I\,\beta^3\, e^{\beta x}[(2\,B - 2\,A) \cos \beta x + (-2\,A - 2\,B) \sin \beta x] + E\,I\,\beta^3\, e^{-\beta x}[(2\,F + 2\,G) \cos \beta x + (2\,G - 2\,F) \sin \beta x] + E\,I\,w_p'''(x) \quad \ldots\ldots\ldots\ldots (6)$$

For the case of a cylinder, Eqs. 4, 5, and 6 are identical except that E I is replaced by D. The constants of integration A, B, F, and G can be adjusted to satisfy any set of boundary conditions.

If one is concerned only with a single segment of beam or a uniform cylinder having continuous loading, the four constants of integration are obtained by solving four simultaneous equations which satisfy the prescribed boundary conditions. However, if discontinuities which divide the structure into n segments, exist in either the loading or the flexural rigidity (E I, D), it becomes necessary to write a separate expression for the radial displacements in each segment between points of discontinuity. This results in 4 n unknown constants of integration which requires writing 4 n equations to satisfy continuity between segments and the prescribed boundary conditions of the system. Considerable work is involved in setting up and solving such equations; therefore, it is the purpose of this paper to present the complete solution to numerous basic cases involving a uniform beam on an elastic foundation having continuous loadings, along with a method of analysis which makes use of these data in solving the more general discontinuous case described above.

METHOD OF ANALYSIS

The method of analysis described herein, for obtaining discontinuity stresses in beams or cylinders, makes use of an iterative procedure, similar to that introduced by H. Cross, for continuous beams and frames.[3] This iterative procedure, commonly known as moment distribution, consists of first obtaining fixed-end moments in each element resulting from the application of applied loads or resulting from some prescribed joint displacement. While the resulting deformations are compatible, the joints are not in static-moment equilibrium; therefore, it is necessary to allow the joints to rotate one by one until each joint in the structure is in static equilibrium. While the rate of convergence of this procedure is not within the scope of this investigation, the examples included in this paper would indicate that, generally, a more rapid rate of convergence is obtained when applying the method to beams on elastic foundation and to cylinders, than when applying it to continuous beams and frames.

To define the changes in the end moments during this iteration, one must predetermine the following quantities: (1) stiffness factor, K, of each element defined as that moment required at the end of an element to produce a unit rotation of that end, (2) carry-over factor, C, for each element defined as the moment produced at one end of an element resulting from placing a unit moment at the opposite end, and (3) distribution factors, DF, at each joint defined as the fractional parts of an externally applied joint moment carried by each element entering the joint.

The general method of analysis presented herein for determining discontinuity stresses makes use of the above iterative procedure along with the use of the principle of superposition. This general method is outlined in the subsequent steps.

Step No. 1.—

(a) The beam or cylinder is considered as externally restrained against radial displacements and rotations at all points of discontinuity except at the free

[3] "Analysis of Continuous Frames by Distributing Fixed-End Moments," by H. Cross, Proceedings, ASCE, 1930.

ends. Locations of concentrated loadings, such as those produced by stiffening rings around a cylinder, are considered as discontinuity points and are, therefore, restrained in the manner described above.

Distributed loadings which, in the case of cylinders, may be due to temperature changes or internal pressures are placed on the beam or cylinder and the fixed-end shears and moments are determined.

(b) The external-moment restraint at each discontinuity point is now removed by the iterative procedure described previously which results in a balanced set of end moments at each discontinuity point.

(c) Knowing the end moments found in Step 1(b), one must now determine the end shears for each segment. This step is best illustrated by considering a single beam or cylinder segment A B as shown in Fig. 1. For the case of a cylinder, this segment is considered as a flat plate on an elastic foundation where the foundation stiffness is equivalent to the hoop stiffness provided by the cylinder. With the end moments $M_{A\,o}$ and $M_{B\,o}$ for segment A B in Fig. 1(a) known, the unknown end shears $V_{A\,o}$ and $V_{B\,o}$ are determined by separating the total loading on the segment into three separate, but equivalent, loadings as shown in Figs. 1(b), 1(c), and 1(d). This separation is necessary due to the fact the elastic foundation forces on the segment in Fig. 1(a) are unknown.

The fixed-end moments and shears for the segment in Fig. 1(b) are those obtained in Step 1(a).

The moments M_A'' and M_B'' in Figs. 1(b) and 1(c), respectively, are now determined to satisfy the conditions that

$$M_A'' = M_{A\,o} + FEM_A$$

and

$$M_B''' = M_{B\,o} + FEM_B \quad \ldots\ldots\ldots\ldots\ldots\ldots (7)$$

The corresponding end shears are determined next for these cases; thus, the total end shears (Fig. 1(a)) can be determined by superposition as follows:

$$V_{A\,o} = V_A' - V_A'' + V_A'''$$

$$V_{B\,o} = V_B' - V_B'' + V_B''' \quad \ldots\ldots\ldots\ldots\ldots\ldots (8)$$

(d) Force equilibrium considerations at the restrained discontinuity points will now yield the reaction forces necessary, at these points, to prevent the normal displacements.

Step No. 2.—A complete set of stiffness-influence coefficients are now obtained as follows:

(a) Restrain all discontinuity points as in Step 1(a) and produce a unit normal displacement at one of the discontinuity points without allowing any joint rotations.

(b) Remove the external moment restraints for each discontinuity by iteration in the same manner as in Step 1(b). This results in known end moments for each segment of the beam or cylinder, for example, $M_{A\,A}$ and $M_{B\,A}$ in Fig. 2.

(c) The end shears are determined, for this case, in a manner similar to that in Step 1(c), as illustrated in Fig. 2. Using superposition, the algebraic

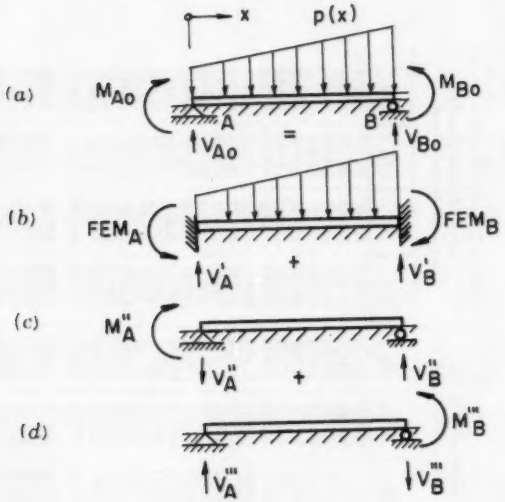


FIG. 1

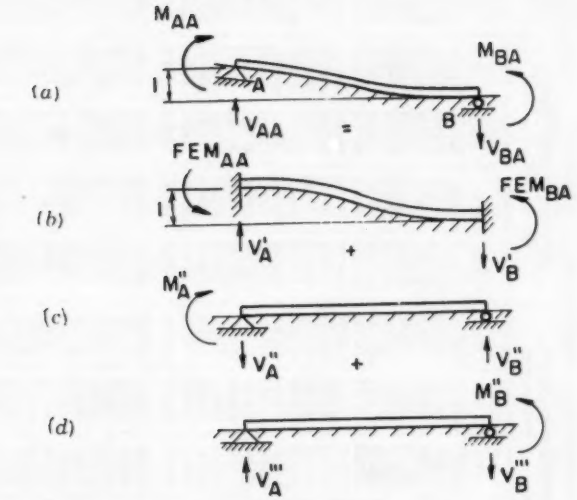


FIG. 2

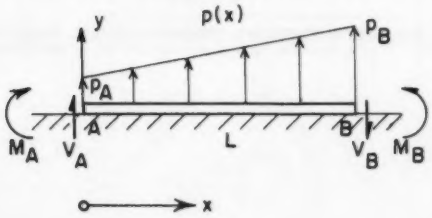
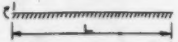
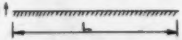
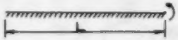
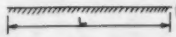


FIG. 3

TABLE 1.—INTEGRATION CONSTANTS

βL																
	A_1	B_1	F_1	G_1	A_2	B_2	F_2	G_2	A_3	B_3	F_3	G_3	A_4	B_4	F_4	G_4
0.05	-227957	-239957	251957	-239957	-112157	-119957	128157	-119957	228057	240057	-252057	240057	-115957	-119957	123957	-119957
0.10	-675155	-749855	825155	-750355	-326355	-373855	426355	-373855	675155	750155	-825155	750155	-350055	-375055	400155	-375055
0.15	-839654	-986654	113655	-988854	-398854	-490154	596354	-490154	839554	987654	-113655	987654	-444454	-493854	543254	-493854
0.20	-187554	-233854	281354	-235054	-875053	-115654	150054	-115654	187554	234454	-281254	234454	-101654	-117254	132854	-117254
0.25	-576353	-764453	960453	-772453	-264053	-376153	520153	-376153	575953	767953	-959853	767953	-320053	-384053	448053	-384053
0.30	-216353	-306253	401653	-311753	-972652	-149753	220753	-149753	216053	308653	-401153	308553	-123453	-154353	185253	-154353
0.35	-930652	-141053	193153	-145153	-410352	-685252	107753	-685252	927452	142753	-192753	142753	-547252	-713852	880352	-713852
0.40	-441552	-719152	102853	-750452	-190752	-347052	581452	-347052	438852	731652	-102453	731652	-268452	-366052	463652	-366052
0.45	-225352	-396252	591752	-420952	-952551	-189852	339252	-189852	223052	405752	-588552	405852	-142152	-203052	264052	-203052
0.50	-121652	-231952	362152	-251952	-502151	-110352	210352	-110352	119552	239452	-359252	239452	-798851	-119852	159852	-119852
0.55	-684951	-142452	232952	-159052	-275751	-672451	137052	-672451	666351	148452	-230252	148452	-470951	-743751	101752	-743751
0.60	-398751	-910551	156152	-104952	-156051	-426651	928651	-426651	382051	959251	-153652	959251	-288451	-481051	673551	-481051
0.65	-237951	-601551	108352	-719851	-901550	-279751	651351	-279751	222851	641551	-106052	641551	-182351	-322051	461751	-322051
0.70	-144551	-408551	774551	-510551	-527150	-188551	470251	-188551	130851	441751	-752551	441751	-116351	-222051	325851	-222051
0.75	-887650	-284051	568151	-372951	-309050	-130151	347951	-130151	762850	311851	-547351	311851	-783450	-157051	235751	-157051
0.80	-547550	-201451	426251	-279551	-179450	-916650	263051	-916650	433250	224951	-406451	224951	-528050	-113551	174251	-113551
0.85	-336550	-145351	326251	-214551	-101550	-657050	202571	-657050	231350	165351	-307451	165351	-360750	-836150	131151	-836150
0.90	-203850	-106451	254251	-168251	-542149	-478150	158851	-478150	106950	123551	-236251	123551	-249050	-626550	100451	-626550
0.95	-119850	-789750	201451	-134451	-255449	-352650	126351	-352650	302749	935750	-184151	935750	-173250	-476550	779950	-476550
1.00	-664649	-592550	161951	-109251	-829048	-263050	101851	-263050	-164849	718250	-145351	718250	-120950	-367350	613850	-367350
1.05	-326349	-448950	131951	-902450	188848	-198250	830350	-198250	-443149	557450	-115951	557450	-843949	-286550	488750	-286550
1.10	-114049	-343050	108851	-756250	765848	-150750	684850	-150750	-600949	436950	-933950	436950	-586849	-225950	393150	-225950
1.15	166148	-264050	907850	-642150	106849	-115550	570550	-115550	-681649	345450	-759050	345450	-404049	-179850	319250	-179850
1.20	939948	-204650	765750	-551850	120049	-891549	479650	-891549	-713349	275250	-621750	275250	-273249	-144450	261450	-144450
1.25	136849	-159350	652450	-479350	122749	-692149	406750	-692149	-713949	220750	-512750	220750	-179249	-116850	215750	-116850
1.30	157249	-124750	561050	-420650	119349	-540149	347550	-540149	-695449	178050	-425550	178050	-111549	-952049	179250	-952049
1.35	163349	-979949	486750	-372350	112449	-423449	299150	-423449	-665549	144250	-355050	144250	-627448	-780749	149950	-780749
1.40	160549	-772549	425750	-332450	103749	-333249	259250	-333249	-629249	117450	-297750	117450	-276848	-643949	126050	-643949
1.45	152349	-610749	375250	-298950	942048	-263149	226050	-263149	-589949	958149	-250650	958149	-263347	-533849	106550	-533849
1.50	141149	-483749	333150	-270650	846848	-208349	198350	-208349	-549749	784049	-211850	784049	150648	-444649	904249	-444649
1.55	128449	-383849	297750	-246550	754948	-165249	174950	-165249	-509949	642649	-179550	642649	273548	-371849	770949	-371849
1.60	115449	-304849	267850	-225850	668748	-131349	155050	-131349	-471549	527249	-152650	527249	356448	-312149	659849	-312149
1.65	102549	-242249	242450	-207950	589248	-104549	138150	-104549	-434849	432449	-130050	432449	409848	-262849	566649	-262849
1.70	902848	-192549	220550	-192350	516948	-831848	123650	-831848	-400249	354349	-110950	354349	441648	-221949	488049	-221949
1.75	788748	-153049	201750	-178650	451748	-662448	111150	-662448	-367849	289749	-947149	289749	457348	-187949	421449	-187949
1.80	684048	-121549	185550	-166550	393548	-527448	100250	-527448	-337649	236049	-809749	236049	461348	-159349	364849	-159349
1.85	589348	-963548	171350	-155750	341848	-419748	907849	-419748	-309649	191449	-692349	191449	456848	-135449	316549	-135449
1.90	504348	-763248	158850	-146150	296048	-333548	825349	-333548	-283649	154149	-591949	154149	446348	-115249	275049	-115249
1.95	428848	-603448	147850	-137550	255748	-264748	752849	-264748	-259649	123049	-505649	123049	431548	-981148	239449	-981148

2.00	362248	-476248	138150	-129850	220448	-209648	689049	-209648	-237549	969248	-431349	969248	413848	-836048	208649	-836048
2.05	303848	-375048	129550	-122750	189448	-165648	632449	-165648	-217149	751048	-367349	751048	394348	-712648	182049	-712648
2.10	252948	-294648	121850	-116350	162448	-130448	582249	-130448	-198349	568448	-311949	568448	373748	-607448	158849	-607448
2.15	208848	-230948	114950	-110850	138948	-102348	537449	-102348	-180949	415548	-264049	415548	352648	-517448	138749	-517448
2.20	170948	-180648	108650	-105150	118548	-798847	497449	-798847	-165049	287748	-222549	287748	331548	-440448	121249	-440448
2.25	138448	-141048	103050	-100250	100848	-620847	461549	-620847	-150349	181248	-186649	181248	310648	-374348	105949	-374348
2.30	110748	-110048	978249	-956249	855647	-479747	429149	-479747	-136849	926447	-155449	926447	290248	-317648	925448	-317648
2.35	873047	-857647	931349	-914049	724147	-368247	399949	-368247	-124449	193847	-128349	193847	270548	-268948	808248	-268948
2.40	676347	-669847	888249	-874849	610947	-280447	373449	-280447	-113049	-409047	-104849	-409047	251548	-226948	705448	-226948
2.45	512147	-525047	848649	-838249	513947	-211747	349449	-211747	-102549	-901147	-845148	-901147	233448	-190948	615148	-190948
2.50	376347	-414247	812049	-804149	430947	-158147	327549	-158147	-929148	-129948	-669348	-129948	216148	-159848	535848	-159848
2.55	264947	-330047	778249	-772249	360047	-116747	307549	-116747	-840648	-161648	-517348	-161648	199848	-133148	466148	-133148
2.60	174547	-266747	746749	-742349	299847	-848446	289249	-848446	-759548	-186548	-386448	-186548	184448	-110248	404748	-110248
2.65	102047	-219447	717449	-714249	248647	-606446	272449	-606446	-685048	-205648	-273948	-205648	169948	-904647	350848	-904647
2.70	447146	-184447	690049	-687749	205447	-424446	256949	-424446	-616848	-219748	-177448	-219748	156248	-735447	303348	-735447
2.75	266344	-158947	664349	-662749	169047	-289346	242749	-289346	-554248	-229548	-952147	-229548	143548	-590447	261648	-590447
2.80	-334746	-140347	640249	-639249	138347	-190846	229549	-190846	-497048	-235848	-254347	-235848	131648	-466547	224948	-466547
2.85	-582846	-126947	617549	-616849	112747	-120446	217449	-120446	-444748	-239148	334147	-239148	120448	-360847	192648	-360847
2.90	-757246	-117247	596149	-595749	913346	-715445	206149	-715445	-396948	-239848	826647	-239848	110148	-270947	164348	-270947
2.95	-872146	-110247	575949	-575649	735946	-389245	195649	-389245	-353348	-238448	123548	-238448	100548	-194747	139448	-194747
3.00	-939046	-104947	556749	-556649	589246	-183845	185849	-183845	-313548	-235248	157048	-235248	915947	-130547	117748	-130547
3.10	-965846	-973646	521349	-521349	369746	-118344	168249	-118344	-244448	-224948	205348	-224948	757147	-315046	820147	-315046
3.20	-902046	-913146	489249	-489249	224346	-173444	152849	-173444	-187448	-210748	234048	-210748	621847	363746	549147	-363746
3.30	-789546	-851946	460149	-459049	130446	-944644	139349	-944644	-140848	-194348	247848	-194348	507347	810746	345147	810746
3.40	-658046	-784146	434449	-433349	717045	-185545	127349	-185545	-102948	-176848	250748	-176848	410847	108647	193547	108647
3.50	-526145	-709646	409149	-408949	364645	-262345	116749	-262345	-723647	-159048	245548	-159048	330047	123747	826046	123747
3.60	-404546	-630546	386749	-386449	164845	-314045	107249	-314045	-480247	-141448	234948	-141448	262747	129847	324245	129847
3.70	-299046	-550246	366049	-365849	608744	-339445	987848	-339445	-288347	-124648	220548	-124648	207147	129547	-518646	129547
3.80	-211546	-471546	347049	-346749	144344	-342045	911948	-342045	-139447	-108848	203848	-108848	161347	124947	-884546	124947
3.90	-141946	-397246	329449	-329149	362442	-327245	843648	-327245	-258046	-942247	185948	-942247	123947	117547	-111147	117547
4.00	-885545	-329146	313149	-312849	270443	-300645	781948	-300645	588246	-808547	167648	-808547	935646	108447	-123347	108447
4.10	-492045	-268246	297949	-297749	117744	-267245	726048	-267245	119947	-687647	149548	-687647	691346	984846	-127847	984846
4.20	-213845	-215346	283949	-283749	220644	-230845	675448	-230845	162147	-579547	132148	-579547	496246	882846	-126947	882846
4.30	-276344	-170146	270849	-270649	307744	-194645	629348	-194645	189147	-483747	115748	-483747	342046	782446	-122347	782446
4.40	884344	-132346	258549	-258449	367344	-160345	587348	-160345	204347	-399647	100348	-399647	221446	686246	-115147	686246
4.50	152145	-101346	247149	-247049	398344	-129545	549048	-129545	210447	-326247	862847	-326247	128446	596246	-106447	596246
4.60	179845	-764145	236549	-236449	403444	-102545	513948	-102545	209447	-262847	735147	-262847	578345	513346	-968946	513346
4.70	182745	-566645	226849	-226449	388644	-797044	481848	-797044	203447	-208547	620447	-208547	535644	438246	-871046	438246
4.80	170545	-413345	217149	-217149	359744	-607944	452348	-607944	193647	-162447	518347	-162447	-326245	370846	-774246	370846
4.90	149945	-296445	208349	-208349	322144	-455144	425148	-455044	181347	-123547	428347	-123547	-590845	311046	-681146	311046
5.00	125545	-208545	200149	-200049	280444	-334044	400148	-334044	167547	-910346	349547	-910346	-764945	258546	-593546	258546
5.10	101145	-144045	192349	-192249	238244	-240244	377048	-240244	152847	-642346	281347	-642346	-868945	212846	-512546	212846
5.20	780144	-975244	184949	-184949	197944	-169044	355748	-169044	137947	-423446	222647	-423446	-919345	173346	-438646	173346
5.30	583544	-649144	178049	-178049	160944	-115944	335948	-115944	123247	-247046	172647	-247046	-929545	139546	-372046	139546
5.40	408144	-422344	171549	-171549	128344	-773743	317648	-773743	109047	-107046	130447	-107046	-910445	110846	-312746	110846
5.50	275444	-275444	165349	-165349	100444	-499743	300548	-499743	955346	209844	951146	209844	-870445	866645	-260446	866645
5.60	171944	-179344	159449	-159449	770643	-310343	284748	-310343	829546	850745	659346	850745	-816645	664645	-214646	664645
5.70	920343	-118644	153949	-153949	580043	-183443	270048	-183443	713446	146246	420946	146246	-754145	497545	-174946	497545
5.80	327243	-825243	148649	-148649	428243	-101443	256348	-101443	607646	189446	228746	189446	-687045	360545	-140846	360545
5.90	-996341	-617743	143649	-143649	309543	-510542	243548	-510542	512146	217846	764045	217846	-618645	249445	-111746	249445

FIG. 3

TABLE 1.—*Continued*

βL	A_1	B_1	F_1	G_1	A_2	B_2	F_2	G_2	A_3	B_3	F_3	G_3	A_4	B_4	F_4	G_4
6.00	-264243	-535943	138949	-138949	218543	-222342	231548	-222742	426846	234446	-419745	234446	-550945	160345	-871645	160345
6.10	-395043	-480643	134449	-134449	150443	-739641	220348	-739641	351246	241446	-131646	241446	-485845	900044	-665845	900044
6.20	-494643	-453443	130149	-130149	100743	-122441	209848	-122841	285046	241146	-197346	241146	-424345	353844	-495045	353844
6.30	-700843	-439243	126049	-126049	651642	-350939	200048	-350939	227446	235246	-243046	235246	-367145	-617443	-354845	-617443
6.40	-426943	-423043	122149	-122149	404442	-146241	190748	-146241	177846	225146	-272446	225146	-314845	-369444	-240945	-369444
6.50	-352543	-407243	118349	-118349	238542	-383041	182148	-383041	135546	212046	-288646	212046	-267345	-588844	-149545	-588844
6.60	-292243	-379943	114849	-114849	132342	-605641	173948	-605641	997345	197046	-294346	197046	-224845	-737144	-774044	-737144
6.70	-240143	-349243	111449	-111449	653541	-626841	166248	-826841	698645	180946	-291946	180946	-187145	-628544	-214244	-828544
6.80	-182943	-313643	108149	-108149	273741	-978341	159048	-978341	452045	164246	-283246	164246	-154045	-875144	-210544	-875144
6.90	-138843	-285443	105049	-105049	887040	-102542	152248	-102542	251145	147646	-270046	147646	-125145	-887344	523344	-887344
7.00	-922542	-246943	102049	-102049	115740	-104442	145848	-104442	901844	131346	-253546	131346	-100245	-873344	744544	-873344
7.10	-577542	-209342	991948	-991948	183739	-101042	139748	-101042	-363544	115746	-235046	115746	-789144	-840344	891544	-840344
7.20	-432042	-176743	964548	-964548	250640	-936341	134048	-936341	-133445	101046	-215346	101046	-608444	-793844	979144	-793844
7.30	-174242	-154643	938348	-938348	628040	-853141	128548	-853141	-205645	872545	-195146	872545	-456844	-738444	102045	-738444
7.40	-245541	-125243	913148	-913148	782640	-688441	123448	-688441	-256845	746445	-175046	746445	-330844	-677844	102545	-677844
7.50	281241	-913942	888948	-888948	109441	-600341	118548	-600341	-290745	631645	-155446	631645	-227244	-614944	100345	-614944
7.60	102142	-775842	865748	-865748	134841	-514541	113948	-514541	-310545	528245	-136746	528245	-143244	-551744	960244	-551744
7.70	105842	-528842	843348	-843348	152641	-433341	109548	-433341	-318845	435945	-119146	435945	-760643	-490144	904144	-490144
7.80	-487441	-172042	821848	-821848	162841	-358341	105448	-358341	-318145	354445	-102746	354445	-232943	-431144	838944	-431144
7.90	135942	-363642	801248	-801248	166141	-290841	101448	-290841	-310445	283045	-876545	283045	173043	-375644	768544	-375644
8.00	195942	-264842	781348	-781348	127141	-204441	976647	-204441	-297445	221245	-739745	221245	476743	-324144	695944	-324144
8.10	190342	-229642	762148	-762148	123441	-160641	940847	-160641	-280745	168045	-616745	168045	695543	-277044	623544	-277044
8.20	-205941	-411841	743648	-743648	116441	-123141	906847	-123141	-261445	122945	-507145	122945	844743	-234344	553144	-234344
8.30	-262041	-524041	725848	-725848	107141	-918140	874547	-918140	-240545	849244	-410445	849244	937543	-196144	485944	-196144
8.40	143842	-845941	708648	-708648	744040	-608240	843647	-608240	-218945	534344	-325845	534344	985143	-162144	422744	-162144
8.50	-166541	0000	692048	-692048	662040	-429940	814247	-429940	-197245	276744	-252545	276744	997343	-132344	364344	-132344
8.60	370741	0000	676048	-676048	577140	-289240	786147	-289240	-175945	693043	-189745	693043	982343	-106344	310844	-106344
8.70	-252941	-252941	660648	-660648	358340	-189040	759347	-189040	-155345	-942843	-136545	-942843	947043	-838643	262444	-838643
8.80	176841	0000	645748	-645748	302040	-117840	733747	-117840	-135945	-220144	-918444	-220144	897043	-646943	219144	-646943
8.90	255341	-127641	631248	-631248	249840	-653239	709347	-653239	-117645	-313744	-549044	-313744	837243	-484743	180644	-484743
9.00	0000	961440	617348	-617348	202840	-279939	685947	-279939	-100845	-380144	-247844	-380144	771243	-348843	146944	-348843
9.10	0000	-770840	603848	-603848	161640	-254138	663547	-254138	-854144	-423844	-645542	-423844	702243	-236443	117544	-236443
9.20	-686341	686340	590748	-590748	765439	-175039	642147	-175039	-714944	-448844	182744	-448844	632543	-144643	921743	-144643
9.30	-229842	153241	578148	-578148	575739	-722138	621647	-722138	-590244	-458644	327144	-458644	563943	-707342	705343	-707342
9.40	240242	240241	565948	-565948	421939	-104638	602047	-104638	-479644	-456444	433244	-456444	497843	-123442	522543	-123442
9.50	-495342	0000	554048	-554048	300439	226438	583247	226438	-382444	-444744	507044	-444744	435343	328042	369743	328042
9.60	0000	287641	542548	-542548	206939	366236	565147	366238	-297844	-425944	554044	-425944	376943	667342	243543	667343
9.70	538641	269341	531448	-531448	136739	386238	547847	386238	-224944	-401944	578944	-401944	323143	912442	140643	912442
9.80	0000	-251141	520648	-520648	857138	337638	531247	337638	-162844	-374444	586044	-374444	274143	108043	581742	108043
9.90	0000	126741	510248	-510248	497938	256238	515347	256238	-110544	-344744	578944	-344744	229943	118343	-669141	118343
10.00	0000	-671940	500048	-500048	255138	165438	500047	165438	-669843	-314044	560944	-314044	190543	123543	-565242	123543

sum of the moments and shears in Figs. 2(b), (c), and (d) must equal the moments and shears, respectively, in Fig. 2(a). For those segments not adjacent to the point of unit displacement, the fixed-end moments (FEM) will be zero.

(d) The normal forces required at all discontinuity points are now calculated from the known end shears in a manner similar to that in Step 1(d). These forces are the stiffness-influence coefficients.

(e) Steps 2(a) through 2(d) must be repeated for a unit displacement at each of the remaining discontinuities, in order that the complete set of influence coefficients be known. These coefficients, when written in matrix form, produce a symmetrical matrix about the diagonal due to Maxwell's law of reciprocal deflections.

Step No. 3.—Using the information determined in Steps 1 and 2, one can write a separate superposition equation for each discontinuity point, expressing the corresponding applied normal force at each point in terms of the unknown, normal displacements of all discontinuity points. These simultaneous equations, when solved, will yield the magnitude of the normal displacements of all discontinuity points in the beam or cylinder.

Step No. 4.—

(a) The end shears and moments, for each case solved in Step 2 involving a unit displacement at a particular point, are multiplied by the displacement of that particular point as found in Step 3. Using superposition, all resulting end moments and shears obtained in this manner are added algebraically to those found in Step 1; thus, the final end moments and shears for each segment of the beam or cylinder are known.

(b) One can now write the general equations for the normal displacement, slope, moment, and shear along each segment using Eqs. 3, 4, 5, and 6, respectively. The constants of integration A, B, F, and G appearing in these equations are determined by superposition, using the final end moments and shears found in Step 4(a) along with the basic integration constants, found in Table 1, which were determined for the unit loading conditions shown at the head of that table.

To illustrate this step, consider a beam segment A B as shown in Fig. 3. Consider also that the end moments M_A and M_B and that the shears V_A and V_B are the final values determined in Step 4(a). The constants of integration A, B, F, and G for this segment may now be expressed in terms of these known end moments and shears and the appropirate constants in Table 1 as follows:

$$\begin{Bmatrix} A \\ B \\ F \\ G \end{Bmatrix} = \begin{bmatrix} A_1 & A_2 & A_3 & A_4 \\ B_1 & B_2 & B_3 & B_4 \\ F_1 & F_2 & F_3 & F_4 \\ G_1 & G_2 & G_3 & G_4 \end{bmatrix} \begin{Bmatrix} M_A \\ V_A L \\ M_B \\ V_B L \end{Bmatrix} \frac{L^2}{E I} \quad \ldots\ldots\ldots (9)$$

For the positive loading shown in segment A B in Fig. 3, the particular solution $w_p(x)$ appearing in Eq. 3 and its derivatives may be expressed as follows:

$$w_p(x) = -\frac{1}{k_f}\left[\left(1 - \frac{x}{L}\right) p_A + \frac{x}{L} p_B\right] \quad \text{(10a)}$$

$$w'_p(x) = -\frac{1}{k_f L}\left[(p_B - p_A)\right] \quad \text{(10b)}$$

and

$$w''_p(x) = w'''_p(x) = 0 \quad \text{(10c)}$$

In the case of a cylinder, D is substituted for E I in Eqs. 9, and E h/r^2 is substituted for k_f in Eqs. 10. Having the constants of integration as defined by Eqs. 9 and the particular solution to the differential equation with its derivatives as given by Eqs. 10, the normal displacements, slopes, moments, and shears are completely defined in the interval A B as used in this illustration. The trigometric relations shown in Table 2, which appear in Eqs. 3, 4, 5, and 6, have been tabulated to simplify the use of these equations.

The general method of analysis, as outlined in Steps 1-4, can be carried out quite rapidly using the basic data plotted in Figs. 4 through 14. All curves have been plotted over a range of β L between 0 and 10. Also shown in these figures are the approximate relationships for obtaining the ordinates for values of β L greater than 10.

The stiffness and carry-over factors to be used in the iteration procedure are given in Figs. 4, 5, and 6. The fixed-end moments and shears for the case of a unit displacement, as required in Steps 2(a) and 2(e), are presented in Figs. 7, 8, and 9. The fixed-end moments and shears required in Step 1(a) for uniform and triangular loadings are given in Figs. 10 through 14.

Three problems have been completely solved in the following section to illustrate the generality of the method of analysis. Example 1 illustrates its use on a combined footing, Example 2 illustrates its use when stresses in a cylinder are produced by axisymmetrical temperature changes, and Example 3 illustrates its use for the case of axisymmetrical loading on a cylinder. The general method presented herein can also account for partial restraint against both normal displacement and rotation at various points along the beam or cylinder.

SAMPLE PROBLEMS

Example 1.—A combined footing 26 ft long, 48 in. wide, and 24 in. deep is loaded by two concentrated loads, as shown in Fig. 15. The foundation stiffness k_f is assumed constant. This is required to determine the foundation pressures and the bending-moment diagram for the footing.

Step No. 1.—The fixed-end moments and shears are zero.

Step No. 2.—The stiffness coefficients are determined as shown in Fig. 16.

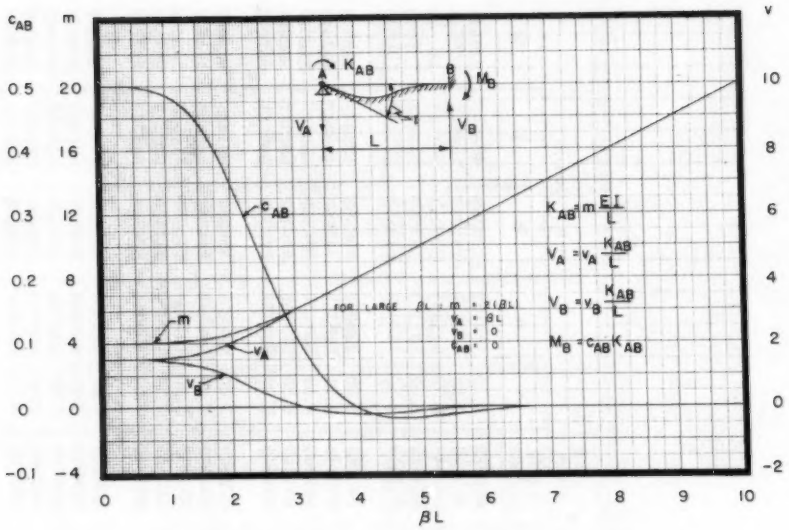


FIG. 4.—BEAM STIFFNESS IN ROTATION - FAR END FIXED

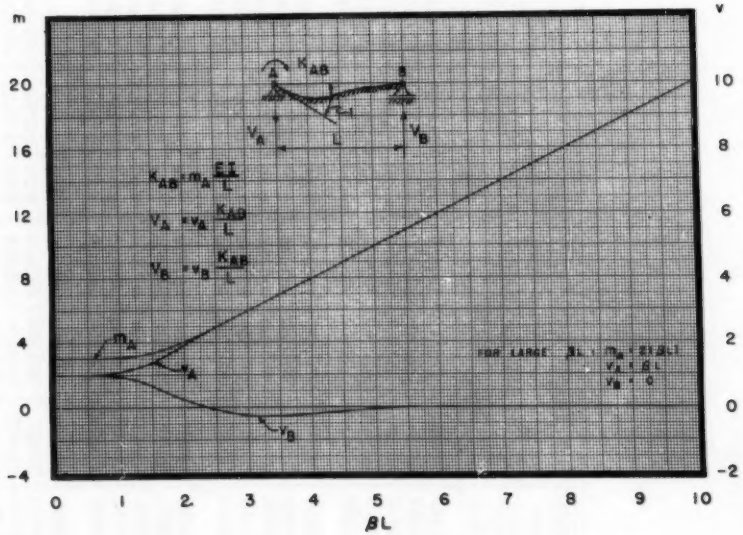


FIG. 5.—BEAM STIFFNESS IN ROTATION - FAR END HINGED

TABLE 2

	$A = e^x \sin x$					$B = e^x \cos x$					$C = e^{-x} \sin x$					$D = e^{-x} \cos x$				
X	A	B	C	D	X	A	B	C	D	X	A	B	C	D	X	A	B	C	D
0.05	525449	104951	475449	950050	2.30	743751	664551	-747649	668049	6.10	812152	-438353	408547	-220548
0.10	110350	109951	903349	900350	2.35	746051	736851	-678549	670149	6.20	409452	-491053	168647	-202248
0.15	173650	114851	128650	851050	2.40	744551	812851	-612749	668949	6.30	915651	544453	308746	183648
0.20	242650	119751	162650	802450	2.45	739051	892551	-550349	664649	6.40	701452	597753	193647	165048
0.25	317650	124451	192650	754550	2.50	729051	975951	-491249	657649	6.50	143053	649553	323447	146848
0.30	398950	128951	218950	707750	2.55	714251	106352	-435449	648149	6.60	229053	698553	423847	129248
0.35	486550	133351	241650	661950	2.60	694051	115352	-382849	636449	6.70	328953	742853	498347	112548
0.40	580950	137451	261050	617450	2.65	668151	124752	-333449	622849	6.80	443653	780553	550347	968347
0.45	682150	141251	277350	574150	2.70	635951	134552	-287249	607549	6.90	573953	809453	582947	822047
0.50	790450	144651	290750	532250	2.75	597051	144552	-243949	590849	7.00	720453	826753	599047	687447
0.55	905950	147751	301550	491850	2.80	550851	154952	-203749	572949	7.10	883453	829653	601447	564847
0.60	102851	150351	309850	452950	2.85	496951	165552	-166249	554049	7.20	106354	814853	592547	454147
0.65	115951	152451	315950	415550	2.90	434851	176452	-131649	534249	7.30	125854	778753	574547	355347
0.70	129751	154051	319950	379850	2.95	363851	187552	-996648	513849	7.40	147054	717453	549347	268047
0.75	144351	154851	321950	345650	3.00	283451	198852	-702548	492849	7.50	169554	626753	518747	191747
0.80	159651	155051	322350	313050	3.10	923050	221752	-187348	450149	7.60	193454	502053	484347	125747
0.85	175751	154451	321150	282050	3.20	143251	-244952	-237948	-406949	7.70	218254	338753	447447	694546
0.90	192651	152851	318450	252750	3.30	427651	-267752	-581848	-364249	7.80	243754	131653	409147	221046
0.95	210351	150451	314550	224950	3.40	765751	-289652	-852848	-322649	7.90	269454	124053	-370347	170546
1.00	228751	146851	309550	198750	3.50	116152	-310152	-105949	-282749	8.00	294954	433753	-331847	488046

1.05	247851	142151	303550	174150	3.60	161952	-328152	-120949	-245049	8.10	319554	802353	-294347	739246
1.10	267751	136251	296650	150950	3.70	214352	-343052	-130949	-209649	8.20	342554	123454	-258347	931546
1.15	288251	129051	289050	129350	3.80	273552	-353552	-136849	-176949	8.30	363054	173554	-224247	107247
1.20	309451	120351	280750	109150	3.90	339752	-358652	-139249	-146949	8.40	380054	230954	-192147	116747
1.25	331251	110051	271850	903449	4.00	413252	-356852	-138649	-119749	8.50	392454	295854	-162447	122447
1.30	353551	981550	262650	729049	4.10	493752	-346852	-135649	-952648	8.60	398854	368654	-135247	124947
1.35	376351	844850	252950	567749	4.20	581252	-326952	-130649	-735148	8.70	397954	449454	-110447	124747
1.40	399651	689250	243050	419149	4.30	675252	-295352	-124349	-543848	8.80	388054	538054	-881646	122247
1.45	423251	513750	232850	282649	4.40	775052	-250352	-116849	-377348	8.90	367354	634554	-683346	118047
1.50	447051	317050	222550	157849	4.50	879952	-189752	-108549	-234148	9.00	333954	738254	508546	112447
1.55	471051	979749	212250	441348	4.60	988552	-111552	-998848	-112748	9.10	285754	848754	-356346	105847
1.60	495051	144650	-201850	589548	4.70	109953	-136251	-909448	-112647	9.20	220554	964854	-225246	984946
1.65	519051	411950	-191450	151949	4.80	121053	-106352	819848	-720047	9.30	136154	108555	-113746	907146
1.70	542851	705250	-181150	235349	4.90	131953	-250452	731548	-138848	9.40	299453	120855	-204945	826946
1.75	566251	102551	-170950	309749	5.00	142353	-420952	646148	-191148	9.50	100354	-133255	-562545	-746446
1.80	589151	137451	-160950	375549	5.10	151853	-619952	564448	-230448	9.60	257354	-145355	-118046	-666946
1.85	611351	175251	-151150	433349	5.20	160153	-849252	487348	-258448	9.70	443454	-157055	-166546	-589746
1.90	632651	216151	-141550	483549	5.30	166753	-111053	415448	-276748	9.80	660854	-167755	-203246	-515946
1.95	652951	260151	-132150	526649	5.40	171053	-140553	349048	-286648	9.90	911854	-177255	-229546	-446146
2.00	671851	307451	-123050	563149	5.50	172653	-173453	288348	-289648	10.0	119855	-184855	-246946	-380946
2.05	689251	358151	-114250	593549	5.60	170753	-209753	233448	-286748					
2.10	704951	412251	-105750	618249	5.70	164553	-249453	184248	-279248					
2.15	718451	469851	-974849	637549	5.80	153453	-292453	140648	-268048					
2.20	729651	531151	-895849	652049	5.90	136453	-338553	102448	-254048					
2.25	738251	595951	-820049	662049	6.00	112753	-387353	692647	-238048					

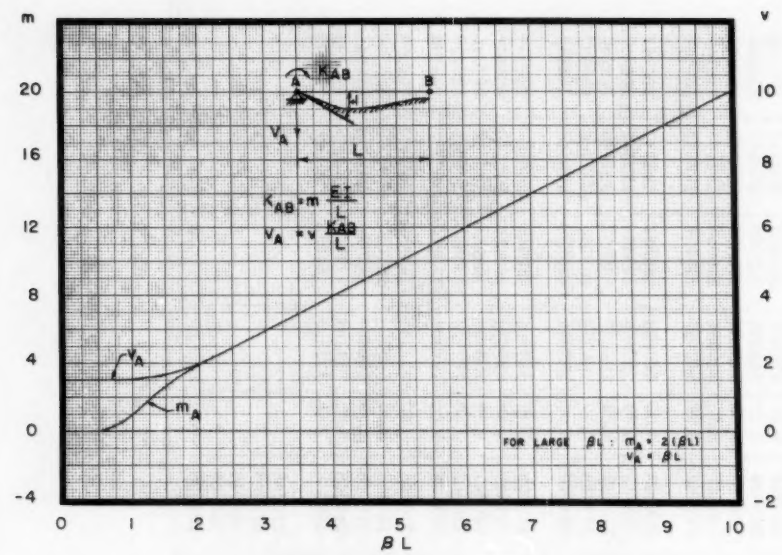


FIG. 6.—BEAM STIFFNESS IN ROTATION - FAR END FREE

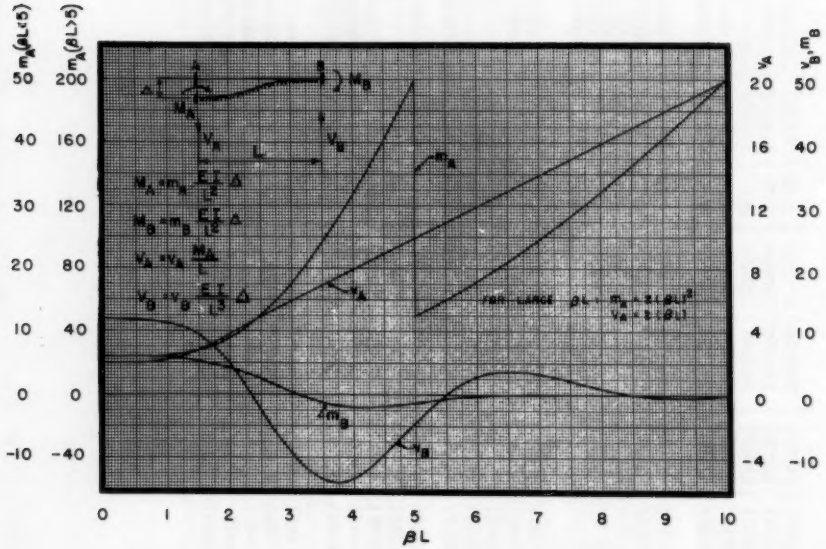


FIG. 7.—BEAM STIFFNESS IN TRANSLATION - FAR END FIXED

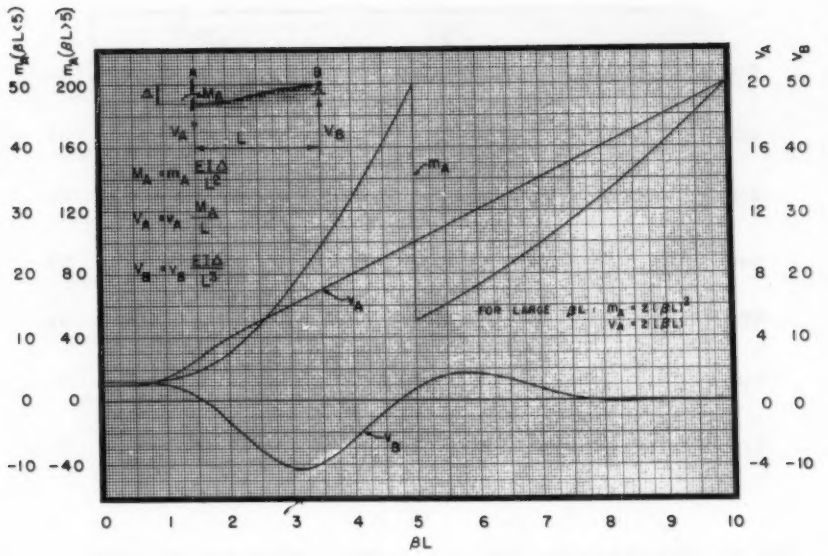

$m_A(\beta L<5)$
$m_A(\beta L>5)$
v_A v_B
50 200 20 50
40 160 16 40
30 120 12 30
20 80 8 20
10 40 4 10
0 0 0 0
-10 -40 -4 -10
$M_A = m_A \frac{EI\Delta}{L^2}$
$V_A = v_A \frac{M_A}{L}$
$V_B = v_B \frac{EI\Delta}{L^2}$
FOR LARGE βL: $m_A = 2(\beta L)^2$ $v_A = 2(\beta L)$
0 1 2 3 4 5 6 7 8 9 10
βL

FIG. 8.—BEAM STIFFNESS IN TRANSLATION - FAR END HINGED

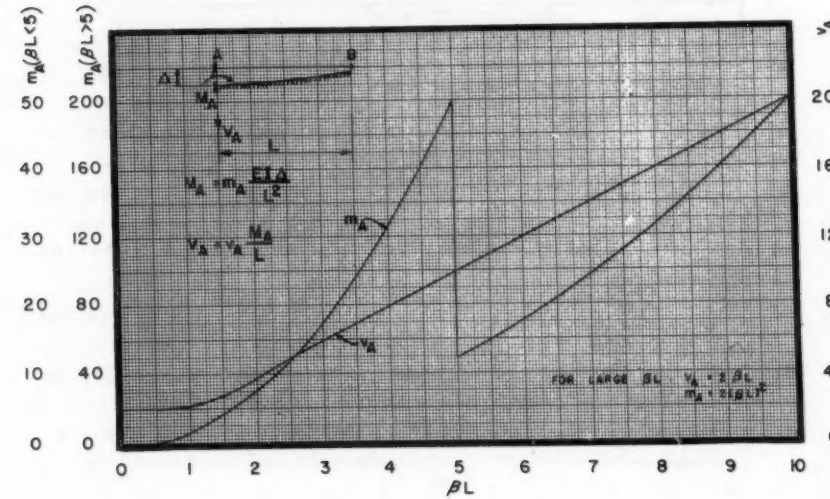

$m_A(\beta L<5)$
$m_A(\beta L>5)$
v_A
50 200 20
40 160 16
30 120 12
20 80 8
10 40 4
0 0 0
$M_A = m_A \frac{EI\Delta}{L^2}$
$V_A = v_A \frac{M_A}{L}$
FOR LARGE βL $v_A = 2\beta L$ $m_A = 2(\beta L)^2$
0 1 2 3 4 5 6 7 8 9 10
βL

FIG. 9.—BEAM STIFFNESS IN TRANSLATION - FAR END FREE

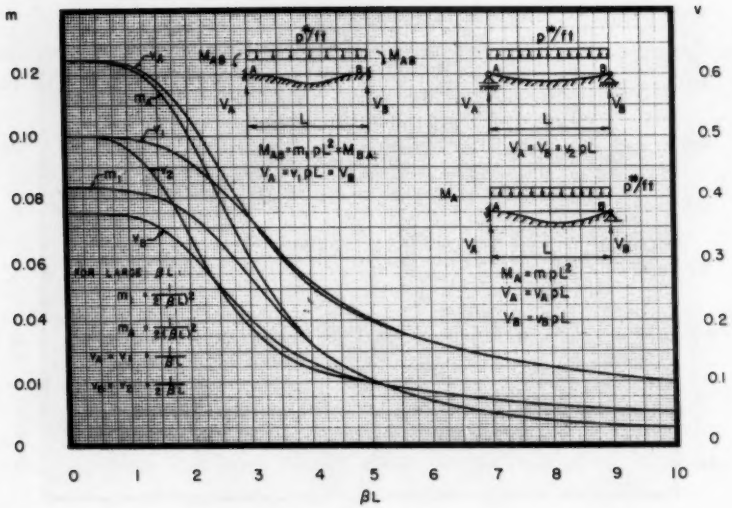


FIG. 10.—UNIFORMLY DISTRIBUTED LOADING - FIXED END MOMENTS AND REACTIONS

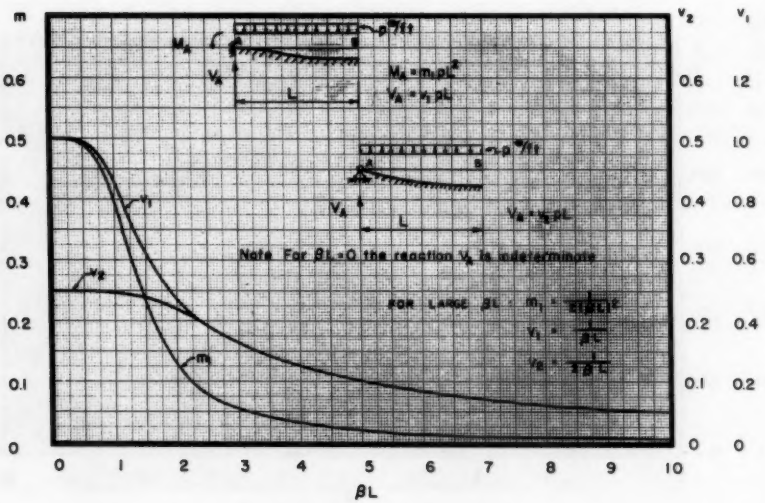


FIG. 11.—UNIFORMLY DISTRIBUTED LOADING - FIXED END MOMENTS AND REACTIONS - ONE END FREE

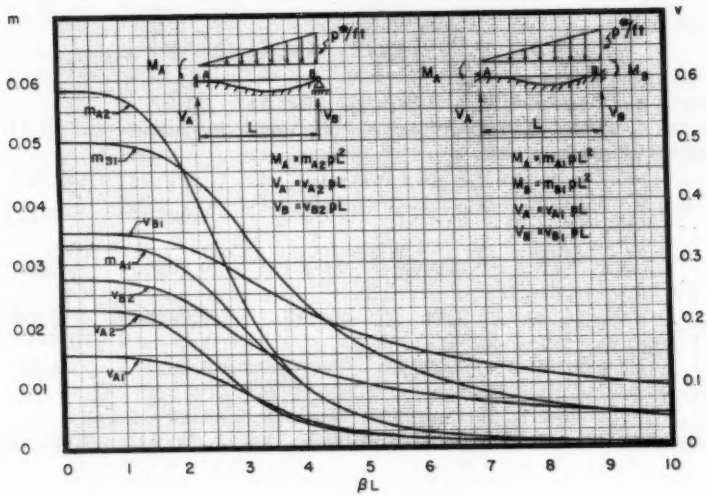


FIG. 12.—TRIANGULAR LOADING - FIXED END MOMENTS - AND REACTIONS

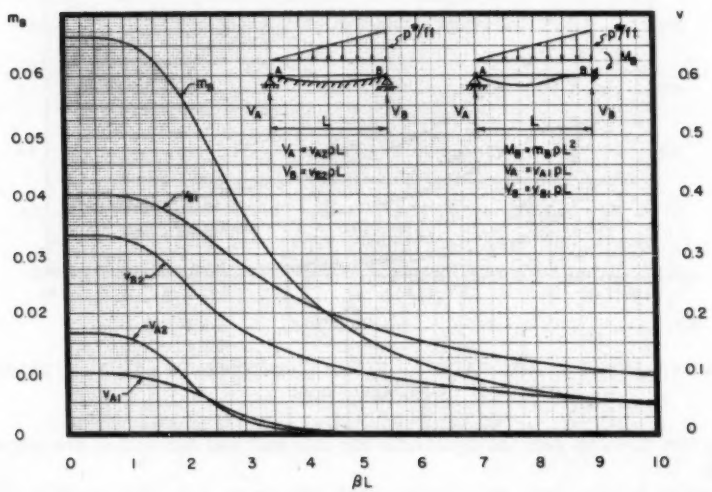


FIG. 13.—TRIANGULAR LOADING - FIXED END MOMENTS AND REACTIONS

Step No. 3.—The superposition equations are

$$0.947\ Y_B + 0.070\ Y_C = -\frac{160}{40}(0.0289)$$

and

$$0.070\ Y_B + 0.339\ Y_C = -\frac{80}{48}(0.0289)$$

from which $Y_B = -0.0928$ in. and $Y_C = - -.1225$ in.

Step No. 4.—The final end moments and shears are shown in Fig. 17. For segment B C Eqs. 9 become

$$\begin{Bmatrix} A \\ B \\ F \\ G \end{Bmatrix} = \begin{bmatrix} 0.000774 & 0.000667 & -0.0118 & 0.00261 \\ -0.000763 & -0.000324 & -0.000107 & -0.00248 \\ 0.0909 & 0.0386 & -0.0116 & 0.00756 \\ -0.0894 & -0.000324 & -0.000107 & -0.00247 \end{bmatrix} \begin{Bmatrix} 0.823 \\ -5.33 \\ 0.0303 \\ 4.07 \end{Bmatrix}$$

which leads to A = 0.0073 in., B = -0.0090 in., F = -0.101 in., and G = -0.0819 in.

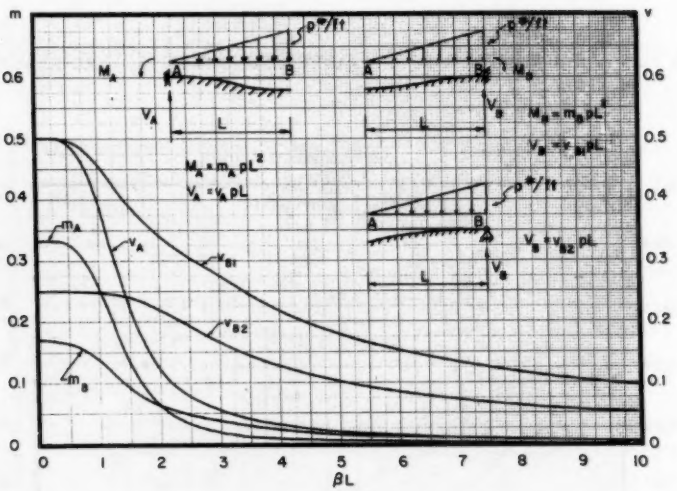
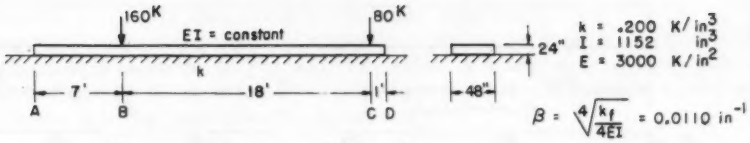


FIG. 14.—TRIANGULAR LOADING - FIXED END MOMENTS AND REACTIONS - ONE END FREE

Step No. 5.—The general equations for moment and deflection are shown in Fig. 18.

Example 2—The long cylindrical shell shown in Fig. 19 is held at a temperature T_1 in region A B and at a temperature T_2 in region C D. It is desireable to calculate the unit stresses in the transition region B C for a linear temperature change over this region. For purposes of analysis, stresses produced



	βL	K	K/EI	C		
BA	.925	.796	EI/L_{AB}	0.114	0	ΣK_B = 0.392 ΣK_C = 0.278
BC, CB	2.375	5.001	EI/L_{BC}	0.278	.258	DF_{BA}= 0,709 DF_{CB}= 0.998
CD	.132	.0005	EI/L_{CD}	0.0005	0	DF_{BC}= 0.291 DF_{CD}= 0.002

FIG. 15

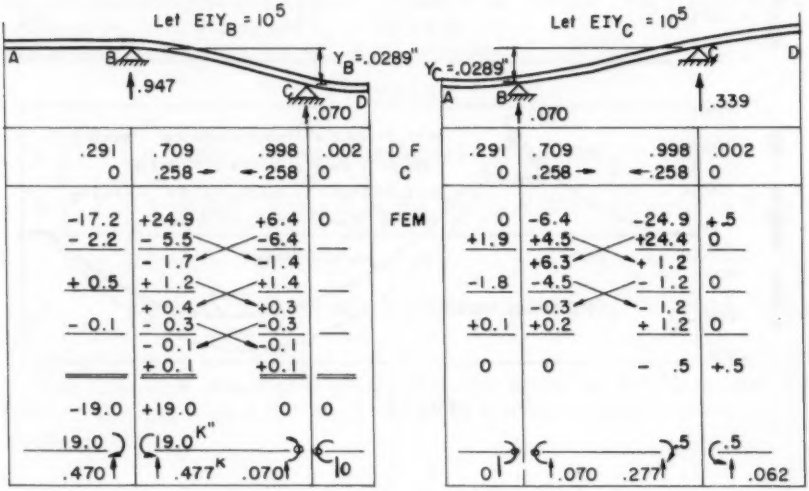


FIG. 16

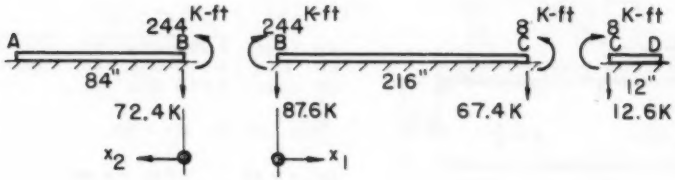


FIG. 17

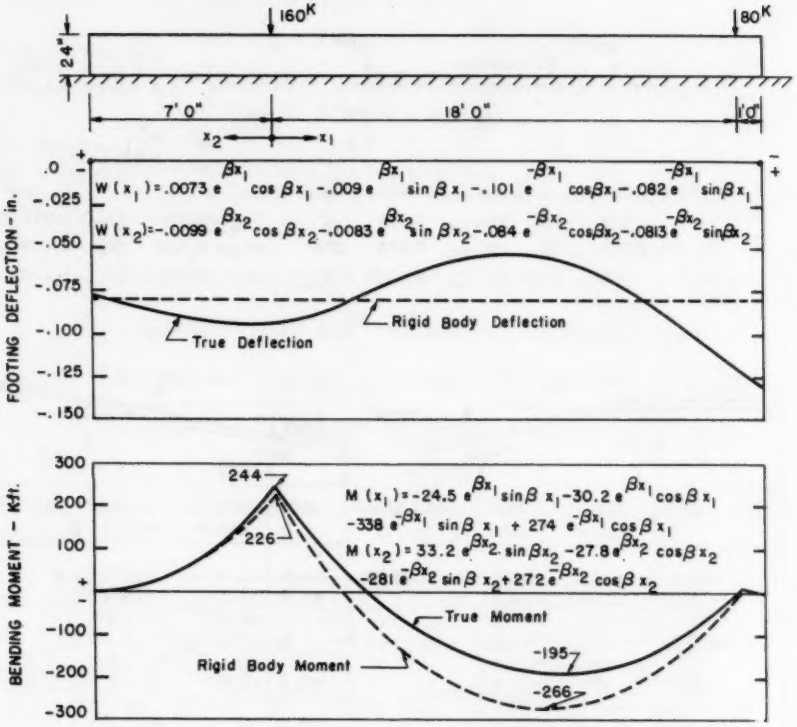
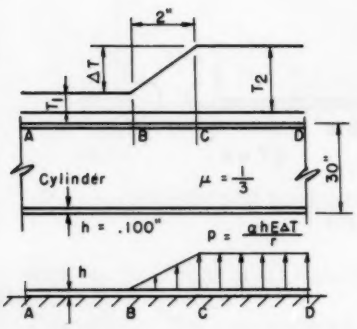

160K
80K
24"
7' 0"
18' 0"
1'0"
x_2
x_1
FOOTING DEFLECTION - in.
$W(x_1) = .0073\,e^{\beta x_1}\cos\beta x_1 - .009\,e^{\beta x_1}\sin\beta x_1 - .101\,e^{-\beta x_1}\cos\beta x_1 - .082\,e^{-\beta x_1}\sin\beta x_1$
$W(x_2) = -.0099\,e^{\beta x_2}\cos\beta x_2 - .0083\,e^{\beta x_2}\sin\beta x_2 - .084\,e^{-\beta x_2}\cos\beta x_2 - .0813\,e^{-\beta x_2}\sin\beta x_2$
True Deflection
Rigid Body Deflection
BENDING MOMENT - K-ft.
244
226
$M(x_1) = -24.5\,e^{\beta x_1}\sin\beta x_1 - 30.2\,e^{\beta x_1}\cos\beta x_1$
$-338\,e^{-\beta x_1}\sin\beta x_1 + 274\,e^{-\beta x_1}\cos\beta x_1$
$M(x_2) = 33.2\,e^{\beta x_2}\sin\beta x_2 - 27.8\,e^{\beta x_2}\cos\beta x_2$
$-281\,e^{-\beta x_2}\sin\beta x_2 + 272\,e^{-\beta x_2}\cos\beta x_2$
True Moment
Rigid Body Moment
-195
-266

FIG. 18



$$\beta = \sqrt[4]{\frac{3(1-\mu^2)}{r^2h^2}} = 1.042$$

	βL	K*	C
AB	Large	2.08 D (Fig. 6)	—
BC	2.08	2.32 D (Fig. 4)	.330 (Fig. 4)
CD	Large	2.08 D (Fig. 6)	—

*For Large βL $K = 2\beta D$

$\Sigma K_B = 4.40$ $\Sigma K_C = 4.40$

$DF_{BA} = .473$ $DF_{CB} = .527$

$DF_{BC} = .527$ $DF_{CD} = .473$

FIG. 19

by the axisymetrical loading, as shown, which are equivalent to those stresses produced by the temperature ΔT across region B C are obtained.

Step No. 1.—In Fig. 20, joints B and C are supported.

Step No. 2.—To determine the stiffness coefficients, the effect of a unit displacement at B is shown in Fig. 21. The effect of a unit displacement of C is shown in Fig. 22; the solution is similar to that for a unit displacement of B.

Step No. 3.—The superposition equations are

$$-9.13\ D\ Y_B + 0.42\ D\ Y_C = -0.186\ p$$

$$0.42\ D\ Y_B - 9.13\ D\ Y_C = -0.1640\ p$$

from which $Y_B = 0.0285\ p/D$ and $Y_C = 0.181\ p/I$.

Step No. 4.—The final end moments and shears are shown in Fig. 23. Since this problem is completely anti-symmetrical, $M_B = -M_C$ and $V_B = V_C$, therefore, the foregoing values should be adjusted as follows: $M_B = 0.052\ p$, $V_B = -0.014\ p$, $M_C = -0.052\ p$, and $V_C = -0.014\ p$.

For section B C ($\beta L = 2.08$), Eq. 9 becomes (Step 4(b))

TABLE 1

$$\begin{Bmatrix} A \\ B \\ F \\ G \end{Bmatrix} = \begin{bmatrix} 0.00273 & 0.00173 & -0.0206 & 0.00382 \\ -0.00327 & -0.00144 & 0.00641 & -0.00649 \\ 0.1248 & 0.0602 & -0.0334 & 0.01680 \\ -0.1189 & -0.00144 & 0.00641 & -0.00649 \end{bmatrix} \begin{Bmatrix} 0.052 \\ -0.014\ L_{BC} \\ -0.052 \\ -0.014\ L_{BC} \end{Bmatrix}$$

Find Maximum Stress at B and C.—Let ΔT represent 200 F, α equal 6.5×10^{-6} in. per in., h be 0.100 in., E equal 30×10^6 psi, and r represent 15 in. These lead to

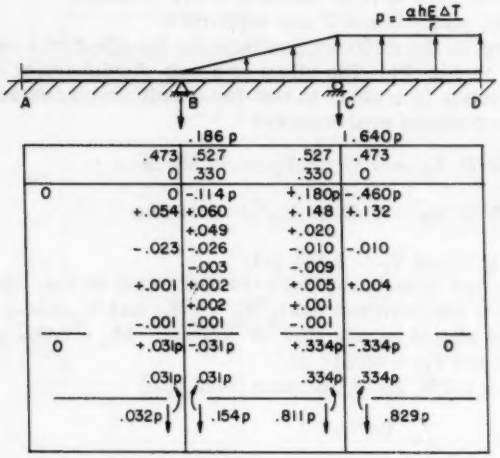
$$p = \frac{2\ h\ E\ \Delta T}{r} = 260 \text{ psi}$$

$$M_B = -\ M_C = (0.052)\ (260) = 13.5 \text{ lb-in. per in.}$$

and

$$\sigma = \frac{6\ M}{h^2}\ 8100 \text{ psi.}$$

Example 3.—The cylinder shown in Fig. 24 has a recess between locations B and C, and is loaded axially on the recess collar at location C. In obtaining radial displacements, slopes, moments, and shears, a longitudinal strip of unit width is removed and placed on an elastic foundation whose stiffness equals



Remarks

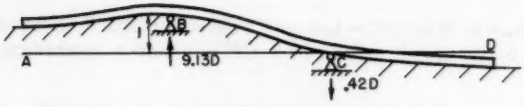
DF; Figs. 4, 6
C; Fig. 4
FEM; Figs. 11, 12 (Step 1a)

Iteration Procedure (Step 1b)

End Moments

End Shears (Step 1c)
Figs. 5, 6, 11, 12

FIG. 20



Remarks

.473	.527	.527	.473
0	.330	.330	0
−2.18 D	+2.40 D	+1.00 D	0
−0.11	−0.11	− .53	−.47
	− .17	− .04	
+ .08	+ .09	+ .02	+.02
	+ .01	+ .03	
	− .01	− .02	−.01
−2.21 D	+2.21 D	+ .46 D	−.46 D
2.21 D	2.21 D	.46 D	.46 D
A — B			C — D
4.57 D	4.56 D	.06D	.48 D

DF
C

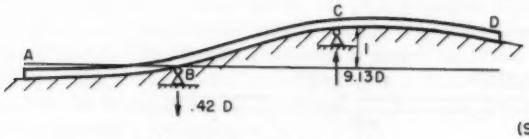
FEM; Fig. 7, 9 (Step 2a)

Iteration Procedure (Step 2b)

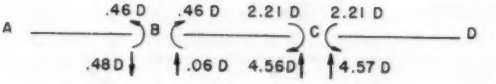
End Moments

End Shears (Step 2c)
Figs. 5, 6, 7, 9

FIG. 21



(Step 2e)



$k_{BB} = +9.13D \quad k_{CB} = -.42D$

$k_{BC} = -.42D \quad k_{CC} = 9.13D$

$$-9.13\,D\,Y_B + .42\,D\,Y_C = -.186p \qquad Y_B = +.0285\,\frac{p}{D}$$

$$+.42\,D\,Y_B - 9.13\,D\,Y_C = -1.640p \qquad Y_C = +.1810\,\frac{p}{D}$$

FIG. 22

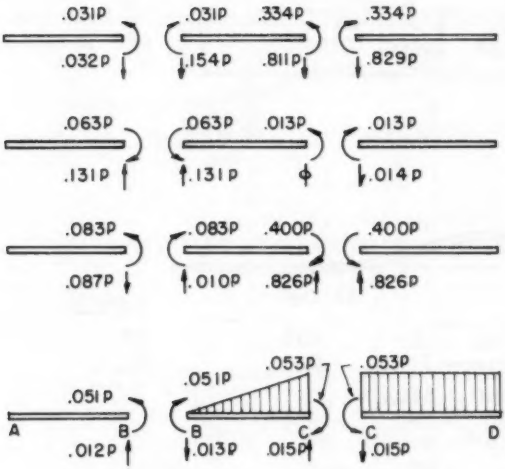
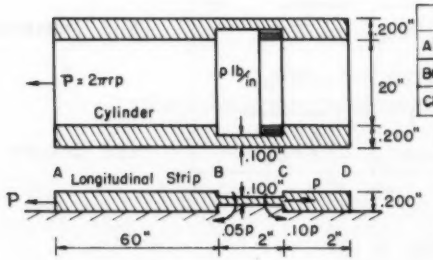


FIG. 23



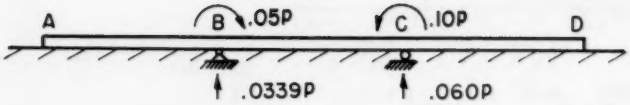
	t	L	r	β	βL	D	K	C
AB	.200	60	10	.901	54.00	$8D_o$	$14.400D_o$	-
BC	.100	2	10	1.275	2.55	D_o	$2.650D_o$	.21
CD	.200	2	10	.901	1.80	$8D_o$	$13.800D_o$	-

$K_B = 17.05 \quad DF_{BA} = .845 \quad DF_{BC} = .155$

$K_C = 16.45 \quad DF_{CB} = .161 \quad DF_{CD} = .839$

$\mu = 1/3$

FIG. 24



	.845	.155	.161	.839		DF
	0	.21	.21	0		C
		.05P	-.10P			Applied Moment
0	-.0422P	-.0078P	+.0161P	+.0839P	0	
		+.0034	-.0016			
	-.0029	-.0005	+.0003	+.0013		Iteration
		+.0001	-.0001			
	-.0001			+.0001		
0	-.0452P	-.0048P	+.0147P	+.0853P	0	End Moments
	.0452P	.0048P	.0147P	.0853P		
	.0407P	.0063P	.0186P	.079P		End Shears

FIG. 25

	.845	.155	.161	.839		DF
	0	.21	.21	0		C
	-12.95D₀	+3.30D₀	+.55D₀			FEM
	+ 8.15	+ 1.50	-.09	-.46		
		- .02	+.31			
	+ .02	0	-.05	-.26		
		- .01				
	+ .01					
0	-4.77D₀	+4.77D₀	+0.72D₀	-.72 D₀	0	End Moments
	4.77D₀	4.77D₀	0.72D₀	.72D₀		
	15.9D₀	5.15 D₀	.20D₀	.67 D₀		End Shears

Remarks

FIG. 26

the hoop stiffness of the cylinder. In this solution, the axis of all segments of the strip are assumed to coincide; therefore, it is necessary to load the strip with concentrated moments at points B and C in order to account for the eccentricities of the axial loads at these points in the cylinder. Once the loading on the unit strip is established, as shown in Fig. 24, the solution is carried out in a manner similar to Example 1.

Step No. 1.—In Fig. 25 joints B and C are supported.

Step No. 2.—To determine the stiffness coefficients, a unit displacement of B is shown in Fig. 26. In Fig. 27 a unit displacement of C is shown.

Step No. 3.—The superposition equations are

$$21.1\ D_o\ Y_B - 0.48\ D_o\ Y_C = -0.0339\ p$$

and

$$-0.47\ D_o\ Y_B + 20.3\ D_o\ Y_C = -0.060\ p$$

from which $Y_B = -0.00167\ p/D_o$ and $Y_C = -0.00299\ p/D_o$.

Step No. 4.—The final end moments and shears are shown in Fig. 28.

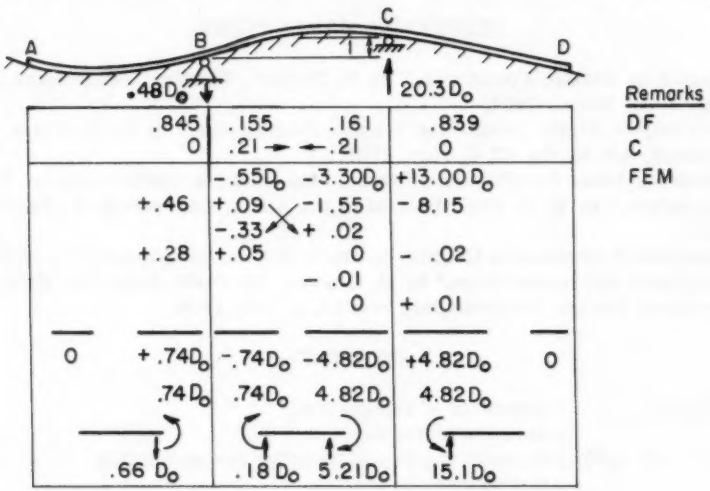


FIG. 27

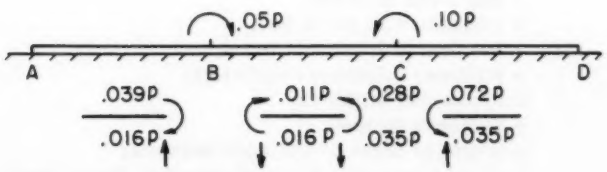


FIG. 28

TABLES

The location of the decimal points in Tables 1 and 2 is given by subtracting 50 from the last two digits of each tabular value. If this difference is zero the decimal point is located in front of the first digit of the number. However, if the difference is +n, the decimal is moved n digits to the right and if the difference is -n, the decimal is moved n digits to the left. For example, the tabular value -1555446 is equal to -.00001555 and the tabular value 358151 is equal to 3.581.

FIGURES

In all figures the flexural rigidity term E I (for beams) should be replaced by the flexural rigidity D when applying to plates.

ACKNOWLEDGMENT

The author wishes to express his appreciation to the Ernest O. Lawrence Radiation Laboratory, Livermore, Calif. for the assistance provided in carrying out the digital computer work required for this paper and also for the assistance provided in preparing the tables and figures included herein.

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2. "Analysis of Finite Beams on Elastic Foundations," by D. C. Gazis. Proceedings, ASCE, No. ST 4, July, 1958.
3. "Influence Lines for Pressure Distribution Under a Finite Beam on Elastic Foundation," by K. C. Ray. Journal of the Amer. Concrete Inst. December, 1958.
4. "Analysis of Continuous Conical Shells of Rotational Symmetry by Method of Successive Approximations," by G. Oravas. Internatl. Assn. for Bridge and Structural Engrg. Publications, Vol. 14, p. 195, 1954.

NOTATION

A, B, F, G	= constants of integration;
C	= carry over factor;
$D = Eh^3/12(1-\mu^2)$	= flexural rigidity of cylinder per unit width;
DF	= distribution factor;
e = 2.718	= Naperian base;
E	= Young's modulus;
FEM	= fixed-end moment;
h	= cylinder wall thickness;
I	= moment of inertia of beam;
k	= stiffness influence coefficient;
k_f	= foundation stiffness;
K	= stiffness factor;
L	= length of beam or cylinder segment;
M	= moment per unit width of cylinder; beam moment;

p = radial loading per unit area for cylinder; load per unit length for beam;

r = cylinder radius;

T = temperature;

V = shear per unit width for cylinder; beam shear;

x = longitudinal distance along beam or cylinder from discontinuity point;

w = radial displacement of cylinder wall; beam deflection;

w_p, w_p', w_p'', w_p''' = particular solution to differential equation and its derivatives;

α = coefficient of thermal expansion;

θ = slope of cylinder wall relative to axis; beam slope;

μ = Poisson's ratio;

$\beta = \sqrt[4]{\frac{3(1-\mu^2)}{r^2 h^2}}$ = for cylinder; and

$\beta = \sqrt[4]{\frac{k_f}{4 E I}}$ = for beam.

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CONTENTS

DISCUSSION

Page

Strength of Rivets and Bolts in Tension, by W. H. Munse, K. S. Petersen and E. Chesson, Jr. (March, 1959. Discussion: September, 1959, October, 1959. Discussion closed.)
by W. H. Munse, K. S. Petersen and E. Chesson, Jr. (closure) 111

Analysis of Two-Column Symmetrical Bents, by B. R. Cooke. (May, 1959. Discussion: May, 1959, October, 1959, December, 1959. Discussion closed.)
by Charles O. Heller (Corrections) 113

Matrix Solution of Beams with Variable Moments of Inertia, by Ming L. Pei. (October, 1959. Prior discussion: January, 1960. Discussion closed.)
by J. Szabó 115
by E. A. Richards. 118

Reinforced Concrete Folded Plate Construction, by Charles S. Whitney, Boyd G. Anderson and Harold Birnbaum. (October, 1959. Prior discussion: None. Discussion closed.)
by C. J. Posey 129

Laterally Deflected Columns, by John Sherman. (October, 1959. Prior discussion: None. Discussion closed.)
by A. A. Eremin. 131
by Benjamin Koo 131

Theory and Test Results on the Fatigue of Metals, by F. Stuessi. (October, 1959. Prior discussion: None. Discussion closed.)
by W. H. Munse 133

Analysis of Continuous Trusses by Carry-Over Moments, by Jan J. Tuma. (December, 1959. Discussion: None. Discussion closes May 1, 1960.)
Corrections. 135

Note.—This paper is a part of the copyrighted Journal of the Structural Division, Proceedings of the American Society of Civil Engineers, Vol. 86, No. ST 3, April, 1960.

Page

The Shell Vault of the Exposition Palace, Paris, by Nicolas Esquillan. (January, 1960. Discussion: None. Discussion closes June 1, 1960.
Corrections. 137

Stability Considerations in the Design of Steel Plate Girders, by Charles E. L. Massonnet. (January, 1960. Discussion: None. Discussion closes June 1, 1960.)
Corrections. 139

New Aspects Concerning Inelastic Instability of Steel Structures, by Bruno Thurlimann. (January, 1960. Discussion: None. Discussion closes June 1, 1960.)
Corrections. 141

STRENGTH OF RIVETS AND BOLTS IN TENSION[a]

Closure by W. H. Munse, K. S. Petersen and E. Chesson, Jr.

W. H. MUNSE,[1] M. ASCE, K. S. PETERSEN,[2] A.M. ASCE and E. CHESSON, Jr.,[3] M. ASCE.—The writers are grateful to Messrs. R. T. Douty, W. R. Penman and E. F. Ball for their contributions and discussions on the behavior of connections with rivets and bolts in tension. Both discussions correctly suggest that stripping failures of the nuts are less likely now that the proof loads of the A.S.T.M. Specification A 325-58T have been increased for heavy hexagon series nuts. This increase was requested by the Research Council on Riveted and Bolted Structural Joints because the results of various laboratory studies, including the authors' investigation, and field experience had indicated that nut stripping was a possibility under the specification in use at the time the tests were conducted. The paper, printed in March, 1959, was presented in February, 1958 and was based on a study reported to the Research Council on Riveted and Bolted Structural Joints in November, 1956.

Messrs. Penman and Ball suggest that only a few structural specifications do not sanction the use of rivets or bolts in connections subject to tension. The AASHO specification[4] states in Section 1.6.26, "Rivets in direct tension should, in general, not be used." The section then goes on to give an allowable stress formula for rivets under combined tension and shear. The AREA specification[5] does not refer to the use of rivets or bolts in tension nor does it provide allowable working stresses for fasteners so loaded. The other major specification[6] widely used in the United States provides an allowable tensile stress on the nominal diameter of the fasteners equal to that permitted for A-7 steel in tension.

Messrs. Penman and Ball also correctly indicate that a rivet rarely completely fills a rivet hole. Nevertheless, the area of a 13/16-in. rivet hole is 0.518 sq in. and, if a rivet in such a hole has a clearance of .007 in. (a reasonable value for a rivet with a grip of several inches), the area of the rivet shank will be about 0.500 sq in. Thus, the use of the nominal hole size to determine the strength of a rivet would produce only a small error (4%) in any evaluation of its tensile capacity based on the material properties. In addition,

---

[a] March, 1959, by W. H. Munse, K. S. Petersen and E. Chesson, Jr.

[1] Prof. of Civ. Engrg., Univ. of Illinois, Urbana, Ill.

[2] Struct. Engr., Rust Engrg. Co., Birmingham, Ala., formerly Research Asst. in Civ. Engrg., Univ. of Illinois, Urbana, Ill.

[3] Asst. Prof. of Civ. Engrg., Univ. of Illinois, Urbana, Ill.

[4] "Standard Specifications for Highway Bridges," Amer. Assn. of State Highway Officials, 7th Ed., 1957.

[5] "Specifications for Steel Railway Bridges for Fixed Spans not Exceeding 400 Feet in Length," Amer. Ry. Engrg. Assn., 1958.

[6] "Specification for the Design, Fabrication and Erection of Structural Steel for Building," Amer. Inst. of Steel Constr., 1956.

it should be noted that the statement which the writers questioned, "Lower strength rivets may develop a load carrying capacity approaching that of comparable size high-strength bolts," refers to the load in pounds, not to stress in pounds per square inch. That certain riveted specimens sustained almost as much as similar bolted specimens may be seen readily in Table 3 of the paper. However, in all fairness to the current A-325 specifications, it should be noted that those bolted connections having load capacity no more than or only slightly better than the riveted joints failed by nut stripping. Had the nuts been of the strengths as now specified, we would expect the bolted joints to develop somewhat greater loads than were shown in the paper. In addition, unless the toes of the flanges of connections of the type considered separate completely, thereby removing the prying action in the joint, we can not expect the bolts (or rivets) to reach their full tensile capacity. This point was illustrated in Fig. 8.

Mr. Douty's test results and analysis are a welcome addition to the paper. These results are for members with two lines of fasteners and, as such, should be comparable to the results of several of the tests reported by the authors. A more effective comparison and evaluation, however, would be possible had the ultimate strengths and specimen details of Mr. Douty's tests been reported.

The analysis presented by Mr. Douty provides an approximate extension of an analysis long employed for the design of tension brackets. The amplification of this analysis to include the development of plastic deformation in the flanges provides another step in the analysis of the behavior of such a member. However, this does not provide an indication of the ultimate strength of such members nor does it provide an indication of the behavior of the members with four lines of fasteners. The analysis of this later type of member can be expected to be much more complex.

It should be noted also that in the analysis presented by Mr. Douty a number of simplifying assumptions have been made. Consequently, the computed relationship can only be considered as approximate. For example, it would be more realistic for the prying force shown in Fig. A to be applied at a point someplace between the bolt line and the toe of the flange. From measurements in the writers' tests such has been observed to be the case. Also, it is known that the ductility of the fasteners has an effect on the prying action in the members and the total ultimate strength of the fasteners. These factors, as well as a number of others, should be given further consideration before this analysis is employed in design.

In his discussion, Mr. Douty has suggested that an overstress in the bolts may produce harmful effects on the behavior of tension brackets. It is difficult to see how such an evaluation can be justified in light of the results of many tests wherein high-strength bolts have been torqued to a condition approaching failure but still retained their full tensile strength. Other tests reported by the writers[7] have shown that similar strengths are obtained when various levels of initial tension are placed in such bolts. Consequently, it would seem unwise to draw a conclusion concerning the effect of a high initial tension on the basis of the single test reported by Mr. Douty.

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7 "Strength of Moment Connections Using High Tensile Strength Bolts," by F. W. Schutz, Jr., AISC Natl. Engrg. Conf. Proceedings, 1959, pp. 98-110.

ANALYSIS OF TWO-COLUMN SYMMETRICAL BENTS[a]

Corrections to Discussion by Charles O. Heller

CORRECTIONS TO DISCUSSION BY CHARLES O. HELLER.[b]—On page 125, in the second line of the first paragraph, "one-day" should read "one-bay."

On page 125, in the first line of the fourth paragraph, the word "wished" should read "wishes."

On page 132, in Figs. 3-2 and 3-3, the forces

$$\left(\frac{\Sigma V_k^{(II)}}{2} + \frac{P_k}{2}\right) \quad \text{and} \quad \left(\frac{\Sigma V_j^{(II)}}{2} + \frac{P_j}{2}\right)$$

should read

$$\left(\frac{\Sigma V_k^{(II)}}{2} + P_k\right) \quad \text{and} \quad \left(\frac{\Sigma V_j^{(II)}}{2} + P_j\right)$$

respectively.

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[a] May, 1959, by B. R. Cooke.
[b] December, 1959.

MATRIX SOLUTION OF BEAMS WITH VARIABLE MOMENTS OF INERTIA[a]

Discussion by J. Szabó and E. A. Richards

J. SZABÓ.[1]—Mr. Pei has presented an interesting procedure which has been used for several years in Hungary. Unfortunately he did not know that this problem was first treated by Prof. J. Egerváry (Bulletin of the Institute of Applied Mathematics of the Hungarian Academy of Sciences, 1954). Concerning problems of two dimensions the solution was given in a lecture by J. Szabó at the Congress of the Austrian Mathematicians in Vienna, 1956 and (especially concerning girder grids) in his paper published in 1958 in the German Journal, "Der Stahlbau." In addition, another paper by J. Szabó on the stability of compression bars with variable cross section, computed by means of the matrix calculus, was presented at the Congress of Engineers and Mathematicians in Belgium, 1958.

As to the essential features of the problem, the following has to be mentioned. The expression given by Mr. Pei,

$$[M] = [A] \cdot [Q]$$

provides the exact value of the bending moments at the equidistant points of the beam as an effect of the applied transverse loads in these points.

By the expression

$$[Y] = [C] \cdot [Q]$$

however, an approximately exact result is given automatically for the transverse displacement. There is no reason why the displacements due to loads applied in these points of the beam of constant rigidity between the equidistant points could not be expressed in equally complicated but exact terms. Such expressions may be established, for instance, on the basis of the Clapeyron equations. Using the symbols of Mr. Pei, we have:[b]

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[a] October, 1959, by Ming L. Pei.

[1] Civ. Engr., Dir., Hungarian Scientific Inst. for Struct. Engrg., Budapest, Hungary.

[b] Here we must depart somewhat from the original symbols of Mr. Pei, because there is a minor mistake in his paper which however, has no bearing on the end result. Correctly:

$$\frac{d^2M}{dx^2} = -q \quad \text{and so} \quad \left[B\right] = \frac{1}{h}\begin{bmatrix} 2 & -1 & & & & & \\ -1 & 2 & -1 & & & & \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ & & & -1 & 2 & -1 & \\ & & & & -1 & 2 & \end{bmatrix}$$

$$\left.\begin{aligned}[B]\cdot[M] &= [Q]\\ [B]\cdot[Y] &= [K]\cdot[M]\end{aligned}\right\} \text{ therefore } [M] = [A]\cdot[Q]$$

$$[Y] = [A]\cdot[K]\cdot[M] = [A]\cdot[K]\cdot[A]\cdot[Q]$$

where [K] is also a square matrix of simple structure:

$$[K] = \frac{h}{6\,E\,J}\begin{bmatrix} 2(z_1+z_2) & z_2 & & & \\ z_2 & 2(z_2+z_3) & z_3 & & \\ & z_3 & 2(z_3+z_4) & & \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ & & & & 2(z_{n-4}+z_n) \end{bmatrix}$$

J is here an arbitrary constant moment of inertia and $z_1, z_2, \ldots . z_n$ is the value of relative inertia moments at the respective sections (Fig. S1).

FIG. S1.

$$z_1 = \frac{J}{J_1};\quad z_2 = \frac{J}{J_2};\quad z_3 = \frac{J}{J_3};\quad \ldots\quad z_n = \frac{J}{J_n}$$

If $z_1 = z_2 = \ldots . = z_n = 1$ (the case of constant moment of inertia), then

$$[K] = \frac{h}{6\,E\,J}\begin{bmatrix} 4 & 1 & & & & & \\ 1 & 4 & 1 & & & & \\ & 1 & 4 & 1 & & & \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ & & & & 1 & 4 & 1 \\ & & & & & 1 & 4 \end{bmatrix}$$

Instead of the deflection influence coefficient matrix

$$[C] = [A]\cdot[h/E\,J]\cdot[A]$$

used by Mr. Pei the modified

$$[\tilde{C}] = [A]\cdot[K]\cdot[A]$$

coefficient matrix has to be used:

$$[Y] = [\widetilde{C}] \cdot [Q]$$

The divergence from the result of the 4th example of Mr. Pei is:

$$h = L/10$$

$$z_1 = z_2 = z_3 = z_4 = z_5 = z_6 = z_7 = \frac{J_o}{2\,J_o} = 0{,}5; \quad z_8 = z_9 = z_{10} = \frac{J_o}{J_o} = 1$$

$$[K] = \frac{L}{60\,E\,J_o} \begin{bmatrix} 2 & 0.5 & & & & & & & & \\ 0.5 & 2 & 0.5 & & & & & & & \\ & 0.5 & 2 & 0.5 & & & & & & \\ & & 0.5 & 2 & 0.5 & & & & & \\ & & & 0.5 & 2 & 0.5 & & & & \\ & & & & 0.5 & 2 & 0.5 & & & \\ & & & & & 0.5 & 2 & 0.5 & & \\ & & & & & & 0.5 & 3 & 1 & \\ & & & & & & & 1 & 4 & 1 \\ & & & & & & & & 1 & 4 \end{bmatrix}$$

$$[C] = \frac{L^3}{6.10^5\,E\,J_o} \begin{bmatrix} 837 & 1454 & 1831 & 1998 & 1985 & 1822 & 1539 & 1126 & 593 \\ 1454 & 2668 & 3452 & 3816 & 3820 & 3524 & 2988 & 2192 & 1156 \\ 1831 & 3452 & 4653 & 5274 & 5355 & 4986 & 4257 & 3138 & 1659 \\ 1998 & 3816 & 5274 & 6192 & 6440 & 6088 & 5256 & 3904 & 2072 \\ 1985 & 3820 & 5355 & 6440 & 6925 & 6710 & 5895 & 4430 & 2365 \\ 1822 & 3524 & 4986 & 6088 & 6710 & 6732 & 6084 & 4656 & 2508 \\ 1539 & 2988 & 4257 & 5256 & 5895 & 6084 & 5733 & 4522 & 2471 \\ 1126 & 2192 & 3138 & 3904 & 4430 & 4656 & 4522 & 3748 & 2114 \\ 593 & 1156 & 1659 & 2072 & 2365 & 2508 & 2471 & 2114 & 1277 \end{bmatrix}$$

and beside the load matrix[c] [Q] in the mentioned example, the displacement matrix [Y] is:

[c] Here a regrettable divergence may be seen between Fig. 4 and the text in the manuscript by Mr. Pei. The writer used the data from Fig. 4.

$$[Q] = P\begin{bmatrix}1\\2\\3\\1\\2\\2\\1\\4\\1\end{bmatrix} \qquad [Y] = [\tilde{C}][Q] = \frac{P\,L^3}{10^5\,E\,J_o}\begin{bmatrix}4284\\8094\\11186\\13274\\14230\\14003\\12649\\9771\\5317\end{bmatrix}$$

Having constant moments of inertia the deflection influence coefficient matrix can be solved for different values of n (see paper by J. Szabó in German, "Computation of Bridgetrusses" in Stahlbau, No. 6, 1958).

For example if n = 6

$$[\tilde{C}] = \frac{L^3}{6.6^4\,E\,J}\begin{bmatrix}50 & 76 & 78 & 62 & 34\\ 76 & 128 & 138 & 112 & 62\\ 78 & 138 & 162 & 138 & 78\\ 62 & 112 & 138 & 128 & 76\\ 34 & 62 & 78 & 76 & 50\end{bmatrix}$$

For example, the coefficient for the deflection at the midspan due to a unit load at the midspan is,

$$\tilde{C}_{33} = \frac{162\,L^3}{6.6^4\,E\,J} = \frac{L^3}{48\,E\,J}$$

This is exact.

E. A. RICHARDS.[2]—In this discussion there is offered an extension of the method, to cantilevers and built-in beams, and there is suggested, a modification to the arithmetic procedure which results in enhanced accuracy. A sum check of use to computer programmers is given.

Cantilevers.—In the case of a cantilever supported at end 0, the boundary conditions become:

$$\text{At} \quad x = 0,\ y = 0,\ \text{and at}\ x = 0,\ \frac{dy}{dx} = 0.$$

If the second condition is replaced by the approximation that at x = h, y = 0, then the finite-difference equations give rise to a matrix of the form

[2] English Elec. Co., Ltd.

$$\frac{1}{h}\begin{bmatrix} 1 & & & & & & & & \\ -2 & 1 & & & & & & & \\ 1 & -2 & 1 & & & & & & \\ & 1 & -2 & 1 & & & & & \\ & . & . & . & . & . & . & & \\ & & . & . & . & . & . & & \\ & & & & & & 1 & -2 & 1 \\ & & & & & & & 1 & -2 & 1 \end{bmatrix}$$

This matrix has an exact inverse, which is

$$-h\begin{bmatrix} 1 & & & & & & & & \\ 2 & 1 & & & & & & & \\ 3 & 2 & 1 & & & & & & \\ 4 & 3 & 2 & 1 & & & & & \\ . & . & . & . & . & & & & \\ . & . & . & . & . & . & & & \\ n-1 & . & . & . & . & . & 2 & 1 & \\ n & . & . & . & . & . & 3 & 2 & 1 \end{bmatrix}$$

and which may be used in place of [A] in the expression

$$[Y] = [A]\left[\frac{Mh}{EI}\right] \quad \ldots\ldots\ldots\ldots\ldots\ldots (21)$$

Built-In Beams.—The deflected shape of a built-in beam may be found by adding to the deflections of the corresponding simply supported beam the deflections arising from the imposition of fixing moments. If these fixing moments have not been determined they can be found from the condition that $\frac{dy}{dx} = 0$ at $x = 0$ and at $x = L$.

Example: Uniform Beam with Uniformly Distributed Load.—The finite difference equation which approximates to

$$\frac{d^2y}{dx^2} = \frac{M}{E\ I}$$

is given by Mr. Pei as

$$[Y] = [A]\left[\frac{h}{E\ I}\right][M] \quad \ldots\ldots\ldots\ldots\ldots\ldots (1)$$

can be expressed as

$$[Y] = [A]\left[\frac{h}{E\,I}\right][a\ M_o + b\ M_n + A\ Q]$$

where M_o is a vector which represents a bending moment decreasing uniformly from unity at $x = 0$ to zero at $x = L$; M_n is a similar vector which is zero at $x = 0$ and unity at $x = L$; a and b are the values of the fixing moments which are to be determined; and A Q is the moment distribution due to the applied load.

In the case of a uniform beam one may rearrange the order of the matrices operating on Q, and write

$$[Y] = \left[\frac{h}{E\,I}\right][A][M_o]\ a + \left[\frac{h}{E\,I}\right][A]\,[M_n]\ b + \left[\frac{h}{E\,I}\right][A^2]\ [Q] \quad \ldots(2)$$

The condition that $\frac{dy}{dx} = 0$ at $x = 0$ and at $x = L$ can be expressed by a finite difference equation, for example[3]

$$\left.\begin{aligned} &(\text{at } x = 0) \quad 4\ y_1 - y_2 = 0 \\ &(\text{at } x = n) \quad 4\ y_{n-1} - y_{n-2} = 0 \end{aligned}\right\} \quad \ldots\ldots\ldots\ldots(3)$$

Taking 10 subdivisions, and performing the matrix multiplications, it is found that

$$\left[\frac{h}{E\,I}\right][A][M_o] = \left[\frac{h}{E\,I}\right][A]\begin{bmatrix} 0.9 \\ 0.8 \\ 0.7 \\ 0.6 \\ 0.5 \\ 0.4 \\ 0.3 \\ 0.2 \\ 0.1 \end{bmatrix} = \frac{h^2}{10^2\,E\,I}\begin{bmatrix} 285 \\ 480 \\ 595 \\ 640 \\ 625 \\ 560 \\ 455 \\ 320 \\ 165 \end{bmatrix} \quad \ldots\ldots (4)$$

3 "Interpolation and Allied Tables," H.M.S.O., London, p. 63.

$$\left[\frac{h}{E\,I}\right][A]\,[M_n] = \left[\frac{h}{E\,I}\right][A] \begin{bmatrix} 0.1 \\ 0.2 \\ 0.3 \\ 0.4 \\ 0.5 \\ 0.6 \\ 0.7 \\ 0.8 \\ 0.9 \end{bmatrix} = \frac{h^2}{10^2\,E\,I} \begin{bmatrix} 165 \\ 320 \\ 455 \\ 560 \\ 625 \\ 640 \\ 595 \\ 480 \\ 285 \end{bmatrix} \quad \ldots\ldots \quad (5)$$

$$\left[\frac{h}{E\,I}\,A^2\right][Q] = \frac{h}{E\,I}\left][A^2\right] \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} = \frac{h\,L^2}{10^4\,E\,I} \begin{bmatrix} 4125 \\ 7800 \\ 10625 \\ 12500 \\ 13125 \\ 12500 \\ 10625 \\ 7800 \\ 4125 \end{bmatrix} \quad \ldots\ldots \quad (6)$$

By substituting the expressions on the right of Eqs. 4, 5, and 6 into the vector Eq. 2, and picking out the first two rows of the vectors, the equations to determine y_1 and y_2 are

$$\left.\begin{aligned} \frac{10^2\,E\,I}{h^2}\,y_1 &= 165\,a + 285\,b + \frac{4125\,L^2}{10^2\,h} \\ \frac{10^2\,E\,I}{h^2}\,y_2 &= 320\,a + 480\,b + \frac{7800\,L^2}{10^2\,h} \end{aligned}\right\} \quad \ldots\ldots\ldots \quad (7)$$

Substituting Eqs. 7 into the first of Eqs. 3 now gives

$$340\,a + 660\,b + \frac{8700\,L^2}{10^2\,h} = 0 \quad \ldots\ldots\ldots\ldots \quad (8a)$$

Similarly, the last two rows of the vector Eq. 2 are substituted into the second of Eqs. 3, giving

$$660\,a + 340\,b + \frac{8700\,L^2}{10^2\,h} = 0 \quad \ldots\ldots\ldots\ldots (8b)$$

From Eqs. 8

$$a = b = \frac{-\,8.7\,L^2}{10^2\,h}$$

which gives the fixing moment at each end. (The true fixing moment is $\frac{W\,L}{12}$ or $\frac{8.333\,L^2}{10^2 h}$, a difference of about 4-1/4%.) Now substituting the calculated value of a and b into Eq. 2, we find the deflections.

$$[Y] = \frac{-\,8.7\,L^2}{10^2\,h}\left[\frac{h}{E\,I}\right][A]\begin{bmatrix}1\\1\\1\\1\\1\\1\\1\\1\\1\end{bmatrix} + \left[\frac{h}{E\,I}\right][A^2]\,[Q]$$

$$= \frac{h\,L^2}{10^4\,E\,I}\begin{bmatrix}-3915 + 4125\\-6960 + 7800\\-9135 + 10625\\-10440 + 12500\\-10875 + 13125\\-10440 + 12500\\-9135 + 10625\\-6960 + 7800\\-3915 + 4125\end{bmatrix} = \frac{h\,L^2}{10^4\,E\,I}\begin{bmatrix}210\\840\\1490\\2060\\2250\\2060\\1490\\840\\210\end{bmatrix}$$

The true central deflection is $\frac{W\,L^3}{384\,E\,I} = \frac{2604\,h\,L^2}{10^4\,E\,I}$ thus the error has increased to 15%, essentially because the calculated deflection is the difference between two similar deflections, and a small percentage error in the determination of a and b leads to a magnified percentage error in the calculated deflection.

The next section shows how this error may be reduced without increasing the number of subdivisions of the beam.

A Device to Improve the Accuracy.—It is clear that the procedure which gave tolerable errors in the analysis of a simply supported beam leads to a much larger error when applied to a built-in beam.

A way to improve the accuracy of the computation is to take as the finite difference equation for

$$\frac{d^2y}{dx^2} = \frac{M}{E\,I}$$

$$y_{i-1} - 2\,y_i + y_{i+1} = \frac{1}{12}\left[\left(\frac{M}{E\,I}\right)_{i-1} + 10\left(\frac{M}{E\,I}\right)_i + \left(\frac{M}{E\,I}\right)_{i+1}\right]h^2 \quad (i=1 \text{ to } n-1)$$

in place of

$$y_{i-1} - 2\,y_i + y_{i+1} = \left(\frac{M}{E\,I}\right)_i \quad (i=1 \text{ to } n-1)$$

This change has no effect on matrix [B], and so none on matrix [A], and involves an easily computed change to the vector on which [A] operates.

The error associated with the finite difference approximation is reduced from

$$-\frac{1}{12}\,\delta^4 + \frac{1}{90}\,\delta^6 + \ldots$$

to[4]

$$\frac{1}{240}\,\delta^6 - \frac{13}{15120}\,\delta^8 + \ldots$$

Example 2: Deflection Influence Coefficient for a Simply Supported Beam of Constant Inertia.—In order to simplify discussion of the modified coefficients when using the revised equations, the operator (*) when applied to a vector will be taken to mean, "Replace each value by $\frac{5}{6}$ of its original value plus $\frac{1}{12}$ the sum of the original adjacent values."

Then the original equation

$$[Y] = [A]\left[\frac{h}{E\,I}\right][A][Q]$$

becomes

$$[Y] = A\left\{\left[\frac{h}{E\,I}\right][A][Q]*\right\}*$$

4 "Numerical Solution of Two Point Boundary Problems," by C. Fox, Oxford Univ. Press, 1957.

For a unit load at midspan, we have

$$[Q] = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \quad \text{and so} \quad [Q]^* = \begin{bmatrix} 0 \\ 0 \\ 0 \\ \frac{1}{12} \\ \frac{5}{6} \\ \frac{1}{12} \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\left[\frac{h}{E\,I}\right][H][Q]^* = \frac{h}{E\,I}\begin{bmatrix} 5 \\ 10 \\ 15 \\ 20 \\ 24\ 1/6 \\ 20 \\ 15 \\ 10 \\ 5 \end{bmatrix}$$

$$\left\{\left[\frac{h}{E\,I}\right][A][Q]^*\right\}^* = \frac{h}{E\,I}\begin{bmatrix} 5 \\ 10 \\ 15 \\ 19.94 \\ 23.61 \\ 19.94 \\ 15 \\ 10 \\ 5 \end{bmatrix}$$

A further multiplication by the middle row of A gives for the central deflection

$$\frac{2087.75\,h^3}{100\,E\,I} = \frac{L^3}{47.896}\,E\,I$$

which is in error by only 1/4%.

Example 3: Built-in Uniform Beam with Uniformly Distributed Load.—The beam considered in Example 1 is now analyzed using the revised finite difference formula. A more refined finite difference approximation is employed for the end slope deflection.

Since M_o and M_n are linear functions of x

$$[M_o]^* = [M_o]$$

and

$$[M_n]^* = [M_n]$$

In this example $[Q]^* = [Q]$ also.

The moment distribution is given by

$$a\,[M_o]^* + b\,[M_n] + A[Q]^* = -\frac{a\,L^2}{10^2\,h}\begin{bmatrix}1\\1\\1\\1\\1\\1\\1\\1\\1\end{bmatrix} + \frac{L^2}{10^2}\begin{bmatrix}45\\80\\105\\120\\125\\120\\105\\80\\45\end{bmatrix}$$

where we have used the previous values for Q, and replaced b by a (by symmetry) and $[M_o^* + M_n^*]$ by a uniform distribution.

The condition for zero slope at end 0 is now taken as

$$4\,y_1 - 3\,y_2 + \frac{4}{3}\,y_3 - \frac{1}{4}\,y_4 = 0$$

which leads to the equation

$$333.332\,a + 166.665\,b + \frac{4141.63\,L^2}{10^2\,h} = 0$$

by the procedure of Example 1.

A similar equation obtained from requiring zero slope at $x = L$ is

$$166.665\,a + 333.332\,b + \frac{4141.63\,L^2}{10^2\,h} = 0$$

giving $a = b = -\frac{8.243}{10^2\,h}\,L^2$, an error of just over 1%.

Using this value, the moment is found to be

$$a\left[M_o\right]^* + b\left[M_n\right]^* + [A][Q]^* = \frac{L^2}{10^2} \begin{bmatrix} -37.43 \\ -2.43 \\ 22.47 \\ 37.47 \\ 42.47 \\ 37.47 \\ 22.47 \\ -2.43 \\ -37.43 \end{bmatrix}$$

and so

$$\left\{a\left[M_o\right]^* + b\left[M_n\right]^* + [A][Q]^*\right\}^* = \frac{L^2}{10^2} \begin{bmatrix} -38.26 \\ -3.27 \\ 21.64 \\ 36.63 \\ 41.64 \\ 36.63 \\ 21.64 \\ -3.27 \\ -38.26 \end{bmatrix}$$

Multiplying by the middle row of [A], the central deflection is found to be

$$\frac{2707.4\ hL^2}{10^4\ EI}$$

which is in error by about 4%.

A large part of this error can still be ascribed to the inaccuracy in determining a and b, as may be verified by substituting the true value of

$$\frac{8.333\ L^2}{10^2\ h}$$

Any attempt to include more ordinates in the condition for zero end slope risks introducing inaccuracy, particularly if the loading is irregular, and although this could be overcome by introducing closely spaced ordinates near the ends the method then loses its simplicity.

A Sum Check.—The sum of all the elements of matrix [A] can be shown to be

$$\frac{h}{n} \bullet \frac{(n^4 - n^2)}{12}$$

and this can be used as a check when generating the coefficients within a computer.

Conclusions.—Mr. Pei's method seems admirably suited to automatic computation. The acceptable results obtainable with 10 subdivisions suggest that only a slight increase in this number is necessary for an accurate general program.

A significant improvement in accuracy is obtainable by use of the device described above, with little extra computation.

The form of each row of the [A] matrix suggests an influence line for each ordinate, and it would seem possible to extend the method to non-uniform spacing, though the derivation would no longer follow from consideration of matrix [B].

REINFORCED CONCRETE FOLDED PLATE CONSTRUCTION[a]

Discussion by C. J. Posey

C. J. POSEY,[1] F. ASCE.—The idea of folding or pleating this material to gain rigidity is undoubtedly a very old one, but the conscious application of structural theory to design a folded plate structure is apparently recent. An American application which antedates any of those mentioned by the authors was made by Byron J. Lambert in his design for the steel portion of the Iowa Field grandstand, built in 1922. Mr. Lambert was granted U.S. Patent No. 1,452,467 on this construction on April 17, 1923, and a number of other grandstands were built utilizing his patent. The Iowa Field grandstand was removed in 1933 and the folded plate elements reassembled in the Field House and at other locations. They are still in service. Mr. Lambert's idea apparently had its origin in his novel design for the interurban crossing over Iowa Avenue, which he built in 1920. Limited headroom forced him to use Z-sections in what might now be described as a "folded plate" type of cross section, though he did not use the term in describing either this or his later designs.

In 1958 a small reinforced concrete folded plate slab was tested to destruction by O. H. Senler.[2] Concrete thickness was 1 in., span 10 feet and the slant height of the plate elements was 15.2 in. The slope was 45° except for a deviation in one fold due to construction difficulties. Longitudinal reinforcement in the bottom of each plate was three 1/4 in. bars, and transverse reinforcement was 12-gauge wire at 4 in. centers. Fig. P1 shows the folded plate loaded with sand bags totalling more than double the design load. Failure came at 2.8 times the design load. There was little deflection, but the slabs had many tension cracks. Collapse of one-half of the structure was so sudden that the immediate cause could not be ascertained. Fig. P2 shows the slab after one-half had failed, spilling the load.

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[a] October, 1959, by Charles S. Whitney, Boyd G. Anderson, and Harold Birnbaum.

[1] Prof. and Head, Dept. of Civ. Engrg., Univ. of Iowa, Iowa City.

[2] "Design, Construction, and Testing of Folded Plate Roof Models," by O. H. Senler, unpublished master's thesis, Civ. Engrg. Dept., Univ. of Iowa, Iowa City, June, 1958.

FIG. P1.—FOLDED PLATE SLAB LOADED WITH SAND BAGS.

FIG. P2.—THE SLAB AFTER FAILURE OF THE TWO PLATES THAT WERE ON THE FAR SIDE IN FIG. A.

LATERALLY DEFLECTED COLUMNS[a]

Discussion by A. A. Eremin and Benjamin Koo

A. A. EREMIN,[1] M. ASCE.—In this paper the author has developed the tangent formula for analysis of stresses in the laterally deflected column. In his Fig. 6 Mr. Sherman has shown graphs to simplify computation of the column stresses.

The tables and graphs simplifying column-stress computations are usually based on the ratio of column slenderness L/r, which helps to visualize the physical properties of column. The parameter X_r expressed by Mr. Sherman is rather difficult to convert to the ratio L/r.

In order to establish the degree of precision of the tangent column formula, Mr. Sherman compared its results with the secant formula. It shall be noted that the precision of the tangent formula varies with precision of computation of the sidesway force H, which is, generally, computed with the methods of distribution of bending moments by successive approximations, either analytical or graphical. Therefore, the tangent formula cannot be designated as exact, when compared with the secant column formula.

Mr. Sherman developed the tangent column formula for the cases when a column has a hinge at the base, or for the case when the base is fixed. In continuous rigid frames the column ends, generally, are partially restrained by the elastic properties of the adjacent members or foundation material. The partially restrained column base was not considered by Mr. Sherman.

Laterally deflected columns are important to consider in the plastic design methods. Therefore, Mr. Sherman's paper is timely and deserves high credit.

BENJAMIN KOO,[2] M. ASCE.—Mr. Sherman has presented a tool by using differential equations explicitly. However, the moment equations can also be obtained by means of slope deflection equations[3] as follows:

$$M_1 = \pm M_{F\,1\text{-}2} - E\,K\left[\alpha_n \theta_1 + \alpha_f \theta_2 - (\alpha_n + \alpha_f)\,R\right] \quad \ldots\ldots (51)$$

$$M_2 = \pm M_{F\,2\text{-}1} - E\,K\left[\alpha_n \theta_2 + \alpha_f \theta_1 - (\alpha_n + \alpha_f)\,R\right] \quad \ldots\ldots (52)$$

in which $M_{F\,1\text{-}2}$ and $M_{F\,2\text{-}1}$ are fixed end moments, $K = I/L$, $R = \delta/L$,

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[a] October, 1959, by John Sherman.

[1] Assoc. Bridge Engr., Calif. State Bridge Dept., Sacramento, Calif.

[2] Struct. and Foundation Engr., M. H. Treadwell Co., New York, N.Y.

[3] "Buckling of Trusses & Rigid Frames," by Winter, Hsu, Koo and Loh, Cornell Univ. Engrg. Experiment Sta., Bulletin No. 36, 1948.

$$\alpha_n = \frac{u}{2 \sin u (\sin u - u \cos u)} (\sin 2u - 2u \cos 2u)$$

and

$$\alpha_f = \frac{u}{2 \sin u (\sin u - u \cos u)} (2\mu - \sin 2\mu)$$

To illustrate, Eq. 31b from which Eq. 34 is derived, can be obtained by direct substitution in Eqs. 51 and 52. With reference to Fig. 5, when H is the only lateral load, $M_{F\,1\text{-}2}$ and $M_{F\,2\text{-}1} = 0$. For column base at 1 is fixed $\theta_1 = 0$

$$\theta_2 = \frac{M_2}{3P} k^2 L X_r \quad \ldots\ldots\ldots\ldots\ldots\ldots (24a)$$

$$\delta = \frac{1}{P} (M_1 + M_2 - H L)$$

$$M_1 = - E K \left[\alpha_f \theta_2 - (\alpha_n + \alpha_f) \delta / L\right] \quad \ldots\ldots\ldots (53)$$

$$M_2 = - E K \left[\alpha_n \theta_2 - (\alpha_n + \alpha_f) \delta / L\right] \quad \ldots\ldots\ldots (54)$$

Solved simultaneously, and by manipulation of trigonometric functions, the answers are:

$$M_1 = \frac{H L}{2} \frac{\tan u}{u} \frac{1 + \frac{X_r}{3} \frac{2u}{\tan u}}{1 + \frac{X_r}{3} \frac{2u}{\tan 2u}} \quad \ldots\ldots\ldots\ldots (31b)$$

$$M_2 = \frac{H L}{2} \frac{\tan u}{u} \frac{1}{1 + \frac{X_r}{3} \frac{2u}{\tan 2u}} \quad \ldots\ldots\ldots\ldots (55)$$

The magnitude of moments contributes greatly to the safe working stress formula (Eq. 34) and its applicability. When the axial compressive force P is small, the moment produced by it is not large enough to affect the combined maximum moment. Therefore, M_1 predominates. A large P with corresponding large deflection, the moment produced by it is big enough to affect the position of the maximum moment. This moment can be larger than M_1 or any other moment along the column. Thus the column approaches the buckling stage.

THEORY AND TEST RESULTS ON THE FATIGUE OF METALS[a]

Discussion by W. H. Munse

W. H. MUNSE,[1] M. ASCE.—The writer is to be complimented on his development and presentation of a series of general empirical relationships which may be used to explain the behavior of a metal by relating its static, creep, and fatigue properties. Although called a theory of fatigue by Mr. Stuessi, the writer would like to refer to the theory as one of behavior rather than fatigue because it is used to describe the behavior of metals under a great variety of conditions ranging from one cycle of load application to many cycles, at any given temperature, and under short-time loading as well as long-time loading. This may be an attempt to make the theory too general and to read into it too many applications, but, from Mr. Stuessi's example it is apparent that he has considered it to be applicable to many problems. On this basis then, the writer would like to make several comments concerning the theory and also to raise several questions concerning further applications of it.

In his paper, Mr. Stuessi attributes the unusual behavior of 75S-T6 aluminum at a relatively small number of cycles, a rather sizeable effect, to the effects of "cold working." This certainly seems reasonable and is probably the most logical explanation that can be offered. However, the writer cannot help but feel that "cold working" or work hardening may be used in this instance much in the way we often use residual stresses. In fatigue studies, brittle fracture studies, and in many other instances, unusual or unexpected behavior which can not be fully explained often has been related to the existence or effect of residual stresses. In many instances this behavior and its relationship to residual stresses has been justified; however, not in all cases. Therefore, it is hoped that further study and tests of the effects of cold working will show that cold working does in fact play the part assumed by the author.

It is noted in the paper that under loadings which are principally compressive in nature, the theory must be somewhat modified. The writer would like to suggest also that the mode of failure, whether on shear planes or in tension, may be an important factor. In recent tests in which the loadings were entirely in compression[9] it was found that the mode of failure was a function of the type of material, and that the mode of failure was of considerable importance in the determination of the fatigue strength of the material. Some metals, for example, are found to fail in shear in a manner similar to the failure often obtained when a concrete cylinder is tested in compression. However, other materials are found to develop a typical tensile-type fatigue

[a] October, 1959, by F. Stuessi.

[1] Prof. of Civ. Engrg., Univ. of Illinois, Urbana, Ill.

crack, even though the applied load is only compressive. Thus, it would appear that the mechanism by which a material fails is an important factor in explaining its behavior fully.

Although nothing is mentioned in Mr. Stuessi's theory about low temperatures or brittle fracture, it does include and relate the effects of temperature and tensile strength. Consequently, one might question whether the theory could be extended so that it would be applicable also to the case of brittle fracture. Perhaps again, the writer is trying to apply too general an interpretation to the theory; however, the development of a single hypothesis to explain the behavior of metals under all types of loadings would be of tremendous value to the engineering profession and certainly a goal worth seeking.

Lastly, the writer would like to question whether the short-time tensile strength referred to in the paper is in reality a limiting strength, obtainable only by means of rapidly applying a load, or whether it is the usual tensile strength obtained in a standard laboratory coupon-type test. We are all aware of the fact that the speed of loading does affect the strength of a metal. However, our laboratory tests have certainly not reached an upper limit of loading speed. Therefore, if the author's theory provides the limiting strength, at the greatest possible speed of loading, this limiting strength may not be determinable physically but could certainly be an extremely interesting limiting property.

ANALYSIS OF CONTINUOUS TRUSSES BY CARRY-OVER MOMENTS[a]

Corrections

CORRECTIONS.—On page 31, the first line following Eq. 6 reads as "The angular load function τ_{jk}" and should read "The angular load function τ_{jj}.

On page 32 in the first line the sentence reads "The formal identity of Eq. (1) and the Eq. (11) of writer's recent paper[(1)] is apparent" and should read "The formal identity of Eq. (11) with the Eq. (11) of writer's recent paper[(1)] is apparent."

On page 32, paragraph B reads "The carry-over factor r_{ij} is the bending moment at j, due to a unit moment at k is equal to zero" and should read "The carry-over factor r_{ij} is the bending moment at j, due to a unit moment at i, if the moment at k is equal to zero."

On page 45, Tables 10 and 11 in the last column the designations "$M_m = \tau_{10}$" and "$M_m = \tau_{12}$" should read "$\overline{M}_m = \tau_{10}$" and "$\overline{M}_m = \tau_{12}$," respectively.

On page 49 in paragraph d the symbol "M_m" should read "$\overline{M}_m$."

[a] December, 1959, by Jan J. Tuma.

THE SHELL VAULT OF THE EXPOSITION PALACE, PARIS[a]

Corrections

CORRECTIONS.—Page 49, paragraph 3, line 7 - "2-3/4 in." should be "2-3/8 in."

Page 55, paragraph 3, line 2 - Lower case "l" omitted before equality sign.

Page 56, Table 1 - There should be no prime over subscript "c" in third column.

Page 60, caption for Fig. 10 - Change "C" to "D" in "DETERMINED."

Line 2 from bottom - The word "years" has been omitted following the word "two."

Page 61, first line below equations - Change capital "L" to lower case "k" before capital "E."

Page 62, line 3 - Letters "u" and "a" transposed in word "precaution."

[a] January, 1960, by Nicolas Esquillan.

STABILITY CONSIDERATIONS IN THE DESIGN OF STEEL PLATE GIRDERS[a]

Corrections

CORRECTIONS.—Page 72, lines 15 and 17 - Bibliography reference numbers should be (2) and (3) instead of (3) and (4).

Eq. 1 - The letter "t" in the second term of the equation is not a superscript but should be on a line with letter "b."

Page 74, last line - Dr. Stüssi has only one initial "F" for Fritz.

Page 78, table - The value of alpha in the long panels is "1.5" and not "1" as shown.

Page 81, line 7 from bottom - Delete "be" before "reasonably."

Page 82, fourth paragraph from bottom, line 1 - Delete "a" before "rectangular."

[a] January, 1960, by Charles E. L. Massonnet.

NEW ASPECTS CONCERNING INELASTIC INSTABILITY OF STEEL STRUCTURES[a]

Corrections

CORRECTIONS.—Page 102, line 5 - Bibliographical reference should be (1).

Page 110, paragraph 2, line 5 - The Greek letter in the mathematical expression following the word "section," should be a lower case Greek letter xi.

Page 114, fourth line above Eq. 21a - Change "y" to "e" in the word "possible."

[a] January, 1960, by Bruno Thurlimann.

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DESIGN OF PRESTRESSED CONCRETE BEAMS BY COMPUTER

By Joseph J. Bonasia,[1] A.M. ASCE

SYNOPSIS

The economical design of prestressed concrete beams is of special interest to engineers concerned with the development of the present highway program. Several groups, such as the Bureau of Public Roads, the American Association of State Highway Officials, and the Prestressed Concrete Institute, recognized the advantages of prestressed concrete and developed standard I-type sections for use as stringers on simple-span highway bridges.

This paper illustrates how an electronic digital computer is advantageously used to select and design standard posttensioned prestressed concrete beams. Two charts are presented which provide information for the rapid selection of satisfactory beams used as interior stringers in simple span, composite beam highway bridges with AASHO H20-S16-44 live load. A description of the method of obtaining the required prestressing force and its location for the results obtained is also presented.

INTRODUCTION

In 1956 the Bureau of Public Roads (BPR) adopted twenty two cross sections recommended for use as prestressed concrete beams for highway bridges with cast-in-place deck slabs. Eleven of these sections are considered in this paper. Similarly in 1957 the Joint Committee of the American Association of State Highway Officials (AASHO) and the Prestressed Concrete Institute (PCI)

Note.—Discussion open until September 1, 1960. To extend the closing date one month, a written request must be filed with the Executive Secretary, ASCE. This paper is part of the copyrighted Journal of the Structural Division, Proceedings of the American Society of Civil Engineers, Vol. 86, No. ST 4, April, 1960.

[1] Civ. Engr., Lockwood, Kessler & Bartlett, Inc., Cons. Engrs., Syosset, N. Y.

presented four cross sections for use as standard beams which are also considered in this paper. The dimensions of these beams are listed in Table 1.

Having decided to use either set of recommended standard beams, the problem becomes one of selecting the appropriate sizes for the various spans and loading conditions. The graphical charts developed herein facilitate the selection of the beam sizes. Curves representing slab thicknesses of 6, 7, and 8 in. are drawn for each beam. Using these curves, the selection of the proper size beam and maximum beam spacing for any given span length without a prior knowledge of the magnitude and location of the prestressing force is possible.

TABLE 1.—STANDARD SECTIONS[a]

Beam No.	A_c	H	B_t	T_t	T'_t	B_b	T_b	T'_b	B'
(a) AASHO-PCI									
1	276	28	12	4.000	3.000	16	5.0	5.0	6
2	369	36	12	6.000	3.000	18	6.0	6.0	6
3	560	45	16	7.000	4.500	22	7.0	7.5	7
4	789	54	20	8.000	6.000	26	8.0	9.0	8
(b) BPR									
1	296	28	15	4.375	1.125	18	4.0	6.0	6
2	320	32	15	3.375	1.125	18	4.0	6.0	6
3	349	36	16	4.250	1.250	18	4.0	6.0	6
4	377	40	17	4.125	1.375	18	4.0	6.0	6
5	405	44	18	4.000	1.500	18	4.0	6.0	6
6	448	44	15	4.375	1.125	20	6.5	7.0	6
7	481	48	17	4.125	1.375	20	6.5	7.0	6
8	525	54	19	3.875	1.625	20	6.5	7.0	6
9	592	60	22	5.000	1.500	20	6.5	7.0	6
10	724	68	25	4.750	1.750	22	6.0	7.5	7
11	804	78	27	4.625	1.875	22	6.0	7.5	7

[a] Areas are in square inches. Dimensions are in inches.

Also described is a program which may be used to design interior and exterior posttensioned prestressed concrete beams for simple-span highway bridges. It is applicable for use with various methods of prestressing. The program will determine the following for both the interior and exterior beam:

1. Required area of prestressing steel and its eccentricity at midspan.
2. Prestressing force required at the jacks.
3. Initial and final flexure stresses at top and bottom fibers of the beams.
4. Ultimate strength of the beam.
5. Required area and spacing of web reinforcement.
6. Camber and dead-load deflection.

Throughout the paper, appropriate sections of the AASHO "Standard Specifications for Highway Bridges," the BPR "Criteria for Prestressed Concrete

Bridges;" and the Joint ASCE-ACI Report on Prestressed Concrete prepared by Committee 323 are followed. Further assumptions, specifications, and terminology for the various materials are stated.

The two programs are coded for the Bendix G-15D General Purpose Digital Computer. The programming method used is the Intercom interpretive and compiling system, with input and output in floating point. The term "program," used throughout this paper, specifies the complete set of coded instructions required to solve a particular problem and is permanently recorded on paper tape punched by the computer. The accuracy of the input and output is limited to five significant figures.

TERMINOLOGY

a. Bridge and Beam Dimensions:

- A = gross area of concrete beam;
- A_i = area of interior beam;
- A_f = area of fascia (exterior) beam;
- A_S = area of prestressing steel;
- A'_S = area of a prestressing steel unit;
- A_V = area of web reinforcement;
- B = effective width of top flange of composite section;
- B' = width of beam web;
- D = distance from center of exterior beam to edge of sidewalk;
- d = distance from extreme compression fiber to centroid of prestressing force;
- e = distance from centroid of prestressing reinforcement to centroid beam;
- h = total depth of beam;
- I = moment of inertia about centroid of cross section;
- I_c = moment of inertia about centroid of composite section;
- j = ratio of distance between centroid of compression and centroid of tension to the depth d;
- L = length of span, center to center bearing;
- S = stringer (beam) spacing;
- s = longitudinal spacing of web reinforcement;
- T = thickness of cast-in-place slab;
- W = total width of sidewalk;
- Y_t, Y_b = distance top, bottom fiber to centroid of beam;
- Z_t, Z_b = section modulus top, bottom, of beam; and
- Z^c_t, Z^c_b = section modulus top, bottom of composite section.

b. Loads:

- DL = effect of dead load;
- F = effective prestressing force after deduction of all losses;
- F'_j = prestressing force required at jacks for prestressing unit, A'_S;
- F_o = prestressing force at transfer;
- I_L = impact fraction;
- M_G = bending moment due to dead load beam;
- M_L = bending moment resisted by composite section;
- M_S = bending moment due to dead load of cast-in-place slab, wearing surface and diaphragms;
- m_u = required ultimate bending moment;

M_u = ultimate resisting moment;
V_c = shear taken by concrete;
V_D = shear due to dead load;
V_F = shear resisted by prestressing tendons;
V_L = shear due to live load; and
V_u = shear due to specified ultimate load and effect of prestressing.

c. Stresses and Strains;
E_c = flexural modulus of elasticity of concrete;
E_{ci} = flexural modulus of elasticity of concrete at transfer;
E_S = flexural modulus of elasticity of prestressing steel;
f'_c = compressive strength of concrete at 28 days;
f'_{ci} = compressive strength of concrete at time of initial prestressing;
f_{cs} = concrete stress at the centroid of the prestressing steel;
f'_s = ultimate strength of prestressing steel;
f_{se} = effective steel prestress after losses;
f_{sy} = nominal yield point stress of prestressing steel;
f'_y = yield point stress of conventional reinforcing steel;
Δ_1 = deflection at time of prestressing; and
Δ_2 = deflection due to dead load of slab.

d. Friction:
e_n = base of Naperian logarithms;
K = friction wobble coefficient;
T_o = steel stress at jacking end;
T_x = steel stress at any point x;
μ = friction curvature coefficient; and
α = total angular change of prestressing steel profile in radians from jacking end to point x.

SPECIFICATIONS

a. Allowable Concrete Stresses:

Beam concrete	f'_c	=	5000 psi
Slab concrete	f'_{cs}	=	3000 psi
Concrete strength at time of prestressing	f'_{ci}	=	4000 psi
Initial compressive stress	$0.55\ f'_{ci}$	=	2200 psi
Initial tensile stress	$3\sqrt{f'_{ci}}$	=	190 psi
Final compressive stress	$0.40\ f'_c$	=	2000 psi
Final tensile stress		=	0

b. Allowable Steel Stress:

Ultimate stress	f'_s	=	240,000 psi
Nominal yield point stress	fsy	=	215,000 psi
Nominal yield point stress, conventional reinforcement	f'_y	=	33,000 psi

c. Sign Convention:

Compression	(+)
Tension	(-)

FORMULAS FOR BEAM SELECTION

The design procedures for prestressed concrete are well known and assumed familiar to the structural engineer; therefore, only certain principles are stated

herein. For composite construction, no temporary intermediate supports are assumed used during the placing of the beam and pouring of the concrete deck slab. Therefore, the beam alone carries the dead load while the composite section carries the live load and any dead load applied after the deck slab has set.

The stresses in concrete produced by the application of external loads and prestressing are computed by the elastic theory. The stress on any fiber due to an eccentrically located prestressing force equals the some of a uniform stress and a bending stress. Additional stresses are produced by external loads, namely, the dead load of the beam, the cast-in-place deck slab, and the live load. The combined stress acting on any fiber a distance y from the neutral axis of the beam and a distance y_1 from the neutral axis of the composite section is expressed as

$$f = \frac{F}{A} \mp \frac{F\,e\,y}{I} \pm \frac{M_G\,y}{I} \pm \frac{M_S y}{I} \pm \frac{M_L y_1}{I_c} \quad \ldots\ldots\ldots\ldots (1)$$

where e is the distance from the neutral axis of the beam to the prestressing force.

Consider the stress on the bottom fiber of the beam at the time of prestressing. From Fig. 1

$$f = \frac{F_o}{A} + \frac{F_o e}{Z_b} - \frac{M_G}{Z_b} \quad \ldots\ldots\ldots\ldots\ldots\ldots (2)$$

The allowable concrete stress, f, before losses equals 2200 psi. Under working loads and for the cases where f'_c = 5000 psi and f'_{ci} = 4000 psi, the losses due to shrinkage, creep, and elastic shortening are about 15%. Under these conditions the stress does not exceed 0.36 f'_c, that is

$$\frac{F}{A} + \frac{F\,e}{Z_b} - \frac{M_G}{Z_b} = 1{,}800 \quad \ldots\ldots\ldots\ldots\ldots\ldots (3)$$

The final stress on the bottom fiber of the beam due to the design working load is

$$f = \frac{F}{A} + \frac{F\,e}{Z_b} - \frac{M_G}{Z_b} - \frac{M_S}{Z_b} - \frac{M_L}{Z_b^c} \quad \ldots\ldots\ldots\ldots\ldots (4)$$

Let the final stress f equal zero. Substituting Eq. 3 into Eq. 4 yields

$$\frac{M_S}{Z_b} + \frac{M_L}{Z_b^c} = 1{,}800 \quad \ldots\ldots\ldots\ldots\ldots\ldots (5)$$

Eq. 5 is written in a convenient form since it contains terms independent of any prestressing force and its location. Given a span, slab thickness, stringer spacing, and any size beam, it is evident that the proper substitution of values

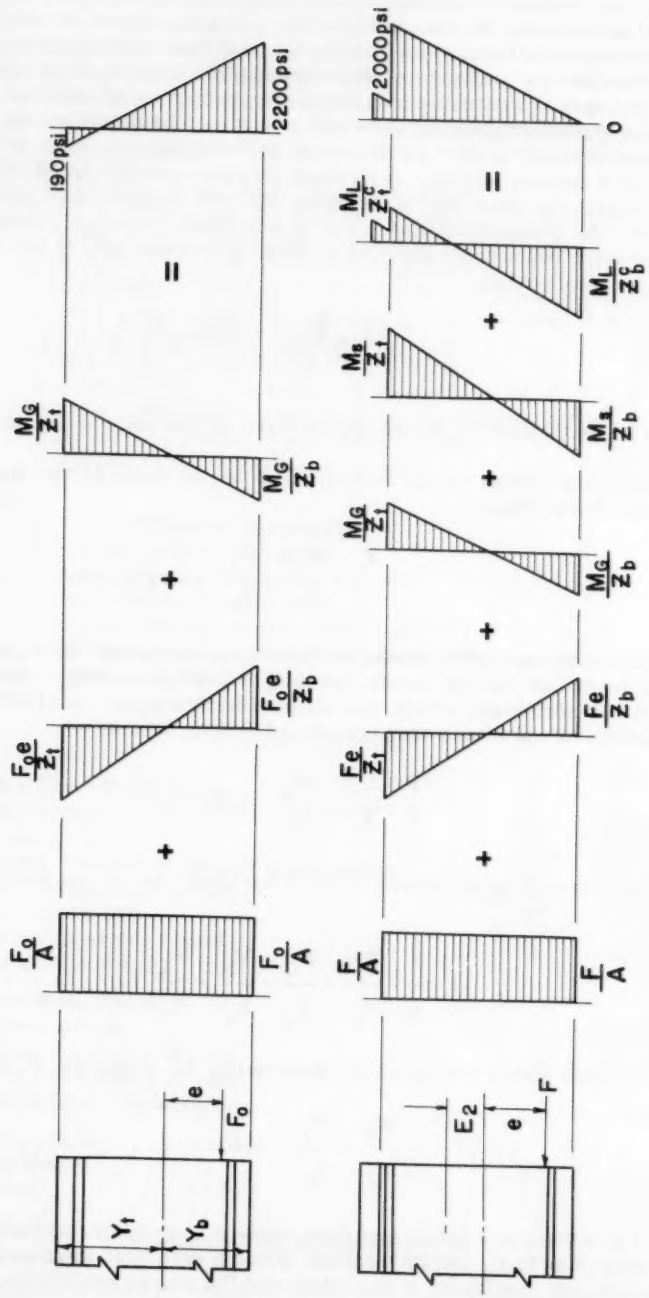

2000psi
0
=
$\frac{M_L}{Z_t^c}$
$\frac{M_L}{Z_b^c}$
+
$\frac{M_s}{Z_t}$
$\frac{M_s}{Z_b}$
+
$\frac{M_G}{Z_t}$
$\frac{M_G}{Z_b}$
+
$\frac{Fe}{Z_t}$
$\frac{Fe}{Z_b}$
+
$\frac{F}{A}$
$\frac{F}{A}$
E_2
e
F
190psi
2200psi
=
$\frac{M_G}{Z_t}$
$\frac{M_G}{Z_b}$
+
$\frac{F_o e}{Z_t}$
$\frac{F_o e}{Z_b}$
+
$\frac{F_o}{A}$
$\frac{F_o}{A}$
e
F_o
Y_t
Y_b

FIG. 1

in Eq. 5 determines the adequacy of that beam in flexure. Negative results indicate that the stresses on the selected beam are within the allowable specified limits; positive results indicate that the selected beam is overstressed.

The terms M_S and M_L in Eq. 5 are readily computed as follows:

Dead Load Moment.—Adding 15% for the dead load of the diaphragms and wearing surface, the maximum bending moment due to the dead load of the slab is

$$M_S = \frac{1.15\ w\ L^2}{8} \quad \ldots\ldots\ldots\ldots\ldots\ldots (6)$$

Live Load Moment.—The maximum live load moment is determined by placing on the span either the standard AASHO H20-S16-44 truck or the lane loading (Fig. 2). By applying to the beam the fraction of a wheel load and the impact fraction in accordance with the AASHO specifications, the design live load moment becomes

$$M_L = \left[\frac{18}{L}(L - 4.67)^2 - 112\right]\left(1 + I_L\right)\frac{S}{10} \quad \ldots\ldots\ldots\ldots (7a)$$

or

$$M_L = \left[L\,(0.08L + 4.5)\right]\left(1 + I_L\right)\frac{S}{10} \quad \ldots\ldots\ldots\ldots (7b)$$

whichever is greater. It is noted that the term L in Eq. 7a must be equal to or greater than 34 ft.

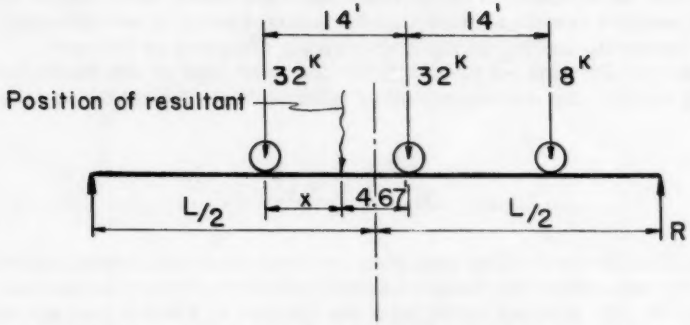
COMPUTER SOLUTION

Eqs. 5, 6, and 7 are readily adaptable for solution by the electronic computer. These equations are utilized to compute for each AASHO and BPR beam the limiting span to the nearest 1/2 ft. The stringer spacings vary from 4 ft 6 in. to 9 ft 0 in. in increments of 6 in. and the slab thickness vary from 6 in. to 8 in. in 1-in. increments. This data allowed the preparation of charts for rapid selection of satisfactory beams.

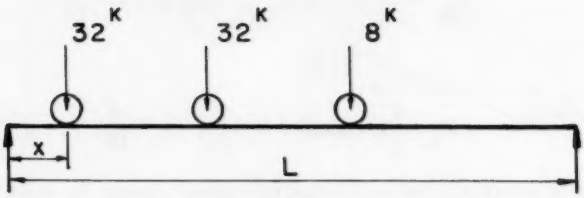
The method used is essentially a trial-and-error method for which the high-speed electronic computer is well suited. Beams are tested for successively increasing spans until the limiting values for each beam are found. A flow chart which traces the various steps followed in the design procedure is shown in Fig. 3. For purposes of this discussion, assume it is desired to obtain the limiting spans for the BPR standard sections using a 6-in. slab.

The program tape contains the necessary physical properties for all the AASHO-PCI and the BPR standard beams. By starting computations at either one of two predetermined locations, the computer will automatically use the chosen set of standard beams (Figs. 3 and 4). Initially, the slab thickness and the stringer spacing are typed by the computer.

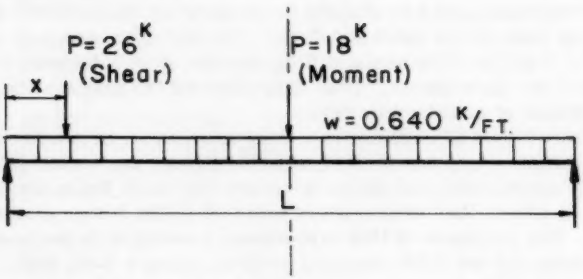
The ability of the computer to make decisions which are based on positive or negative values, is best illustrated by the use of a simple example. AASHO



(*a*) Position of Standard Truck for Maximum Moment



(*b*) Position of Standard Truck for Maximum Shear



(*c*) Lane Loading for Maximum Moment and Shear

FIG. 2

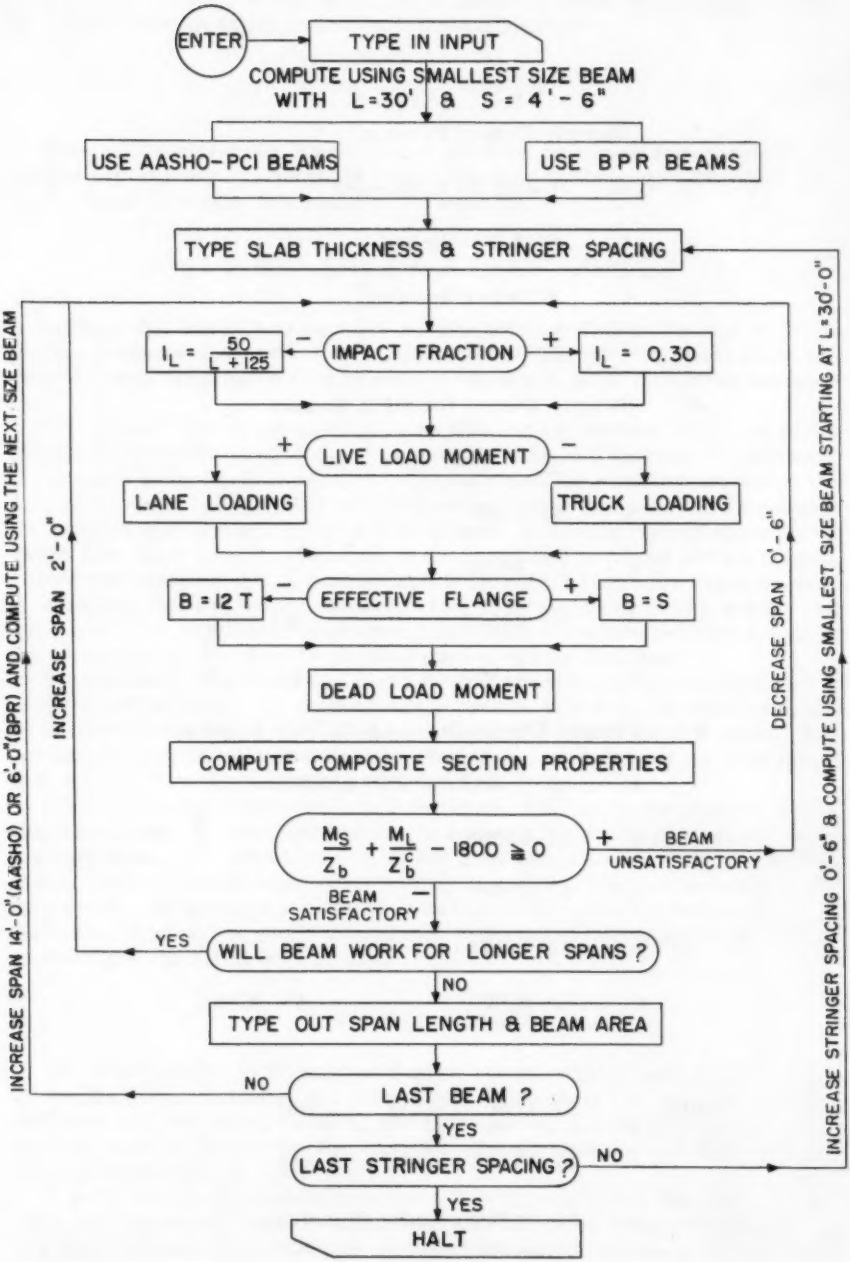


FIG. 3.—FLOW CHART

1. Sample Output Format

   a. Compute Command

   b. Slab Thickness

   c. Stringer Spacing

   d. Span Length Beam Area

2. Sample Output for BPR Beams

   a. qf 102 z s

   b. 51.6000

   c. 51.45000

   d. 52.45500 53.29600

   d. 52.51500 53.32000

3. Sample Output for AASHO-PCI Beams

   a. qf 097 z s

   b. 51.60000

   c. 51.45000

   d. 52.44000 53.27600

   d. 52.6000 53.36900

Note: 51.6000 equals 6.000

52.45500 equals 45.500

53.36900 equals 369.000

FIG. 4

specifies the impact fraction which has a maximum value of 30%. Accordingly the impact equation is written in the following manner:

$$I_L = \frac{50}{L + 125} - 0.30 \geq 0 \quad \ldots\ldots\ldots\ldots\ldots\ldots (8)$$

When the results of Eq. 8 are zero or positive, I_L equals 0.30 and when the results are negative I_L equals the computed value of the impact fraction. Using the same principle, the result of the equation

$$M = \left[L\,(0.08L + 4.5)\right] - \left[\frac{18}{L}\,(L - 4.67)^2 - 112\right] \geq 0 \quad \ldots\ldots\ldots (9)$$

is investigated. Positive or negative results indicate whether the lane or truck loading produces the greater bending moment. The determination of the effective flange width follows the same procedure and needs no further explanation.

The moment due to the cast-in-place slab and the moment of inertia of the composite beam are computed. It is now possible to determine the adequacy of the beam using Eq. 5. A negative value indicates the stresses are within the allowable limits. At thispoint the computer assumes that the beam is suitable for a longer span and will continue to do so until otherwise instructed, as shown on the flow chart (Fig. 3). The span is increased 2 ft at a time and all computations are repeated until a positive result from Eq. 5 shows the beam is overstressed for this and longer spans. In order to attain the desired accuracy, the span length is repeatedly decreased 1/2 ft until a negative solution indicates the computer has reached the limiting span length for that beam.

Summarizing, the computer found BPR Beam No. 1 to be adequate until it reached a 46-ft span. By decreasing the span to 45 ft 6 in., a negative result is obtained from Eq. 5 which indicates the limiting span for that beam. According to previous instructions the computer now types out the span length and, for identification purposes, the beam area (Fig. 4).

If there are additional beams to investigate, the span is increased a specified amount and the next size beam is introduced. Based on preliminary studies, the spans are increased 6 ft for the BPR beams and 14 ft for the AASHO beams, thereby eliminating unnecessary computing. Upon completion of all size beams, the stringer spacing is incremented 6 in. and the entire computation cycle is repeated. At the completion of the computations for all the stringer spacings, the computer will halt.

USE OF THE CHARTS

The charts shown in Figs. 5 and 6 show curves relating span length, stringer spacing, slab thickness, and beam size. The charts are used to determine the beam size for any given span, stringer spacing, and slab thickness. They are also used to determine the maximum stringer spacing and slab thickness for any beam given the span length.

A point falling immediately to the left of a given curve indicates that that beam and any larger beam is satisfactory for the given span, stringer spacing, and slab thickness. For example, consider the AASHO beams shown in Fig. 5. A point lying to the left of the 6-in. slab curve for beam No. 3 and to the right

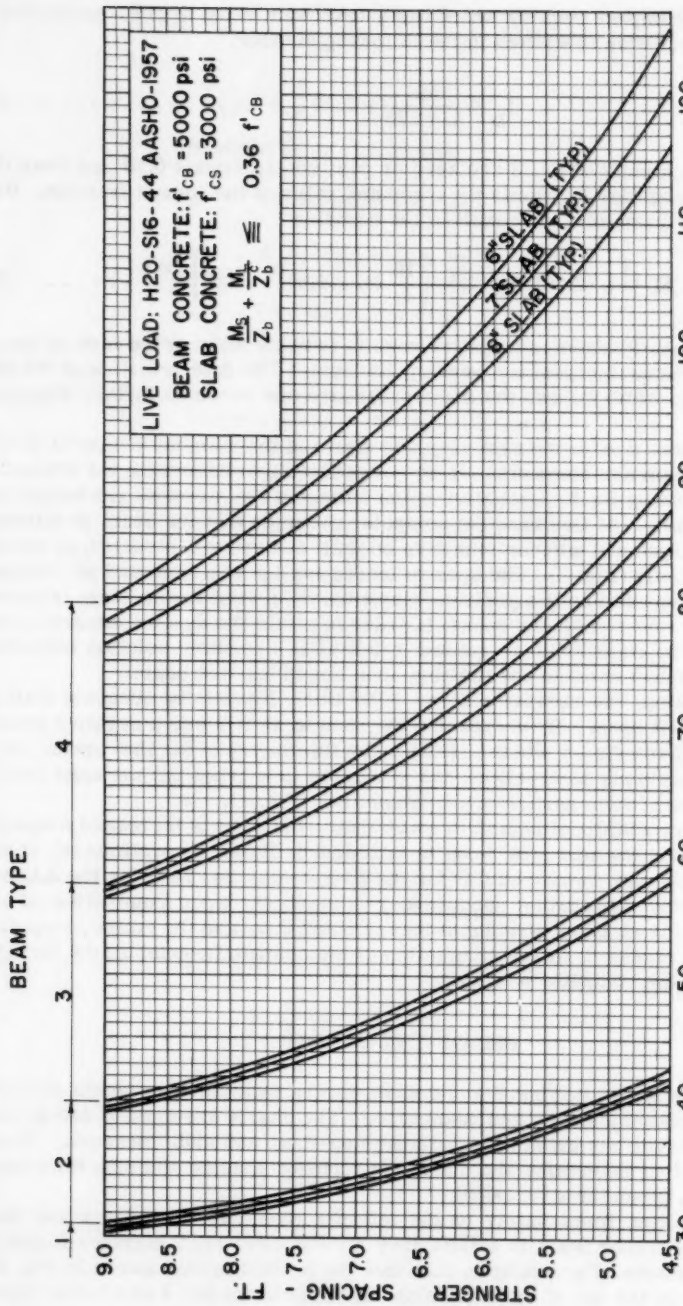


FIG. 5.—PRESTRESSED CONCRETE BRIDGE BEAMS, AASHO-PCI STANDARD SECTIONS, POST-TENSIONED

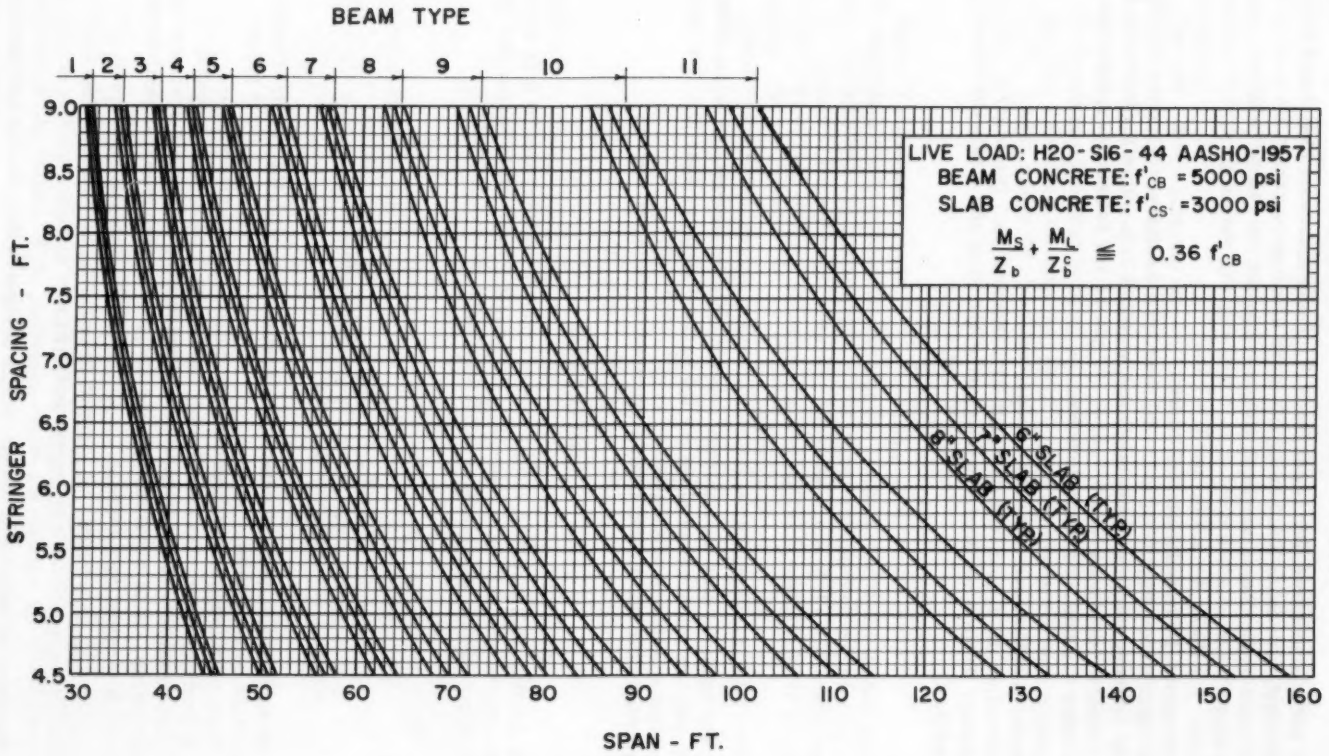


FIG. 6.—PRESTRESSED CONCRETE BRIDGE BEAMS, BPR STANDARD SECTIONS, POST-TENSIONED

of the 6-in. slab curve for beam No. 2 indicates that beam Nos. 3 and 4 are satisfactory, with beam No. 3 being the more economical choice.

The uses of the charts are best illustrated by the following examples:

Example 1.—

Given: L = 60 ft, S = 6.5 ft, and T = 7 in.

Determine: Required size BPR beam

Solution: Enter Fig. 6 with L = 60 ft and S = 6.5 ft and determine that the point of intersection lies immediately to the left of 7-in. curve for beam No. 6. This indicates that beam Nos. 6, 7, 8, 9, 10, and 11 are satisfactory, with beam No. 6 being the most economical choice.

Example 2.—

Given: L = 80 ft

Determine: S, T, and required size BPT beam

Solution: Draw a vertical line through the 80 ft span and obtain the following information:

Beam	T, in inches	Maximum S, in feet
8	7	6.25
8	6	6.50
9	7	7.45
9	6	7.75
10	6, 7, and 8	9.00

On the basis of this information and taking into account the width of the bridge and the thickness of slabs for the various stringer spacings, the most economical solution is determined.

The charts for the AASHO beams are used in a similar manner.

LIMITATIONS

The AASHO-PCI and BPR beams are intended for spans from 30 ft to 100 ft and the charts presented are intended primarily for use in this range. For the longer spans shown, the following conditions must be considered:

1. The equation

$$\frac{F_o}{A} + \frac{F_o e}{Z_b} - \frac{M_G}{Z_b} = 1{,}800 \text{ psi} \quad \ldots\ldots\ldots\ldots\ldots (10)$$

must be satisfied. A large enough prestressing force with the required eccentricity must be available.

2. Live load and/or dead load deflection may become excessive.

3. A depth to span ratio may limit some beams for the longer spans.

4. The suitability of the thinner slabs for the longer stringer spacings was not considered. It is questionable whether a 6 in. slab is economical for the 8 ft and 9 ft stringer spacings.

DESIGN FORMULAS

A program is described herein which designs both interior and exterior posttensioned prestressed concrete beams for simple-span highway bridges

with AASHO H20-S16-44 live load. A method of computing the magnitude and location of the prestressing force is developed for use with the electronic computer. Equations are derived for the loading conditions and the design of the beams. These equations are presented in a form suitable for efficient programming.

Flexure.—The equations for the maximum dead and live load bending moments for the interior beams have been previously derived. These equations, with modifications for the different loading conditions, are used for the exterior beam.

The maximum live load bending moment for the exterior beam is determined by applying to the beam the reaction found by placing a wheel 2 ft from the curb (Fig. 7). Taking moments about the center line of the first interior beam, the fraction of the wheel load becomes

$$\frac{S + D - W - 2.0}{2S}$$

The maximum live load bending moment for the exterior beam is then found by modifying Eq. 7a or 7b accordingly. The loads assumed carried by the composite section of the exterior beam are the dead loads of the sidewalk, railing, and future wearing surface (if any) and the sidewalk and truck (or lane) live loading.

Consider the stress on the bottom fiber of a prestressed concrete beam allowing no tension under working loads

$$\frac{F}{A} + \frac{F\,e}{Z_b} - \frac{M_G}{Z_b} - \frac{M_S}{Z_b} - \frac{M_L}{Z_b^c} + 0 \quad \ldots\ldots\ldots\ldots (11)$$

Letting

$$\frac{M_G}{Z_b} + \frac{M_S}{Z_b} + \frac{M_L}{Z_b^c} = \Sigma f \quad \ldots\ldots\ldots\ldots\ldots\ldots (12)$$

and substituting in Eq. 11

$$\frac{F}{A} + \frac{F\,e}{Z_b} - \Sigma f = 0 \quad \ldots\ldots\ldots\ldots\ldots\ldots (13)$$

Solving for e,

$$e = \frac{Z_b}{F}\left(\Sigma f - \frac{F}{A}\right) \quad \ldots\ldots\ldots\ldots\ldots\ldots (14)$$

Consider a posttensioning system using units (or tendons) of high tensile strength steel bars or wires encased in metal or rubber conduits. For greater economy these units are placed at the maximum distance possible from the centroid of the beam. With provision made for proper concrete cover, the distance to any unit or the centroid of a group of units is known. For example,

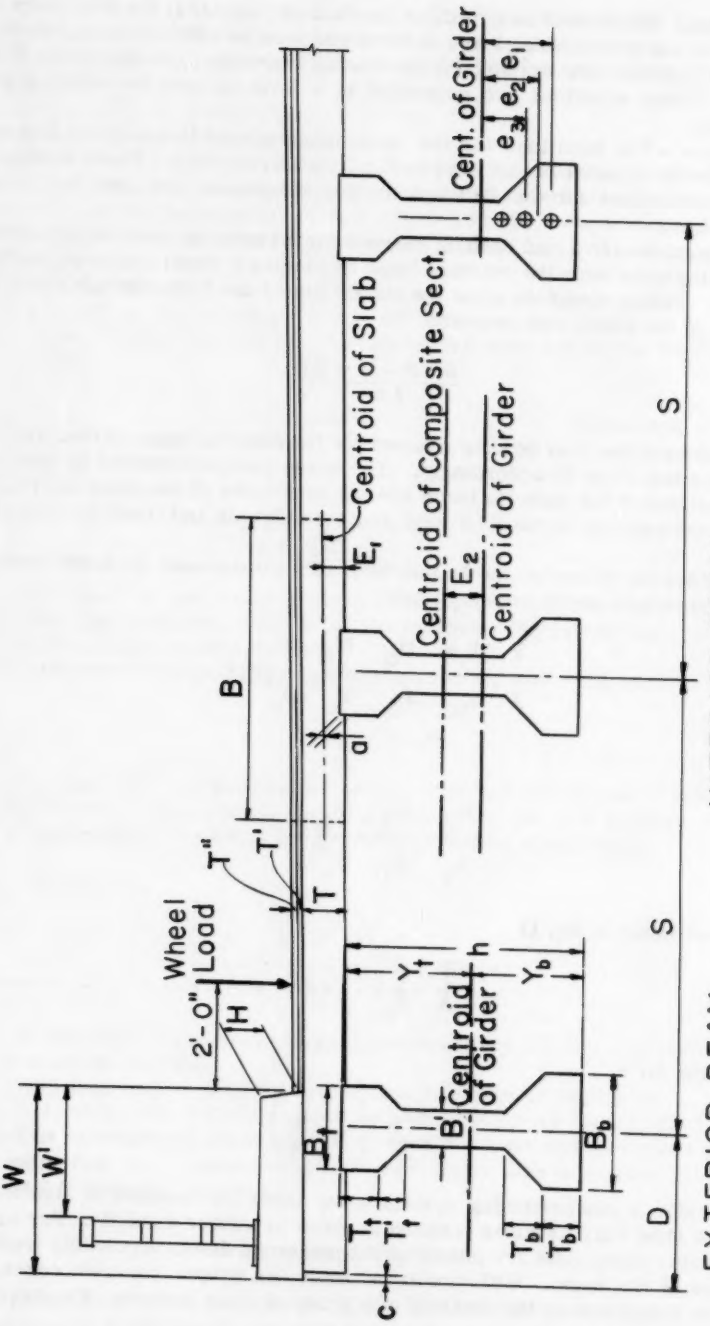

W
W'
Wheel Load
2'-0"
H
T"
T'
T
B
d'
Centroid of Slab
Centroid of Composite Sect.
Centroid of Girder
E1
E2
Cent. of Girder
e3 e2 e1
S
INTERIOR BEAM
EXTERIOR BEAM
Yt
h
Yb
Centroid of Girder
B'
Bt
Bb
Tt
T't
T'b
Tb
D
c

FIG. 7

in Fig. 7, e_1 is the maximum eccentricity (distance) for unit #1; e_2 is the maximum eccentricity for units #1 and #2, etc.

Assume the concrete stresses on the bottom fiber of the beam due to the external loads are known. The effective steel prestress or the allowable design stress is 0.60 f_{sy} or 0.80 f'_s, whichever is smaller. The prestressing force for a unit or group of units having an area A_s, is therefore:

$$F = A_s f_{se} \quad \ldots\ldots\ldots\ldots\ldots\ldots (15)$$

Consider first a unit or group of units having an area A_s and located a distance e from the centroid of the beam. The prestressing force F_1, and e_1, are substituted in Eq. 13, giving

$$\frac{F_1}{A} + \frac{F_1 e_1}{Z_b} - \Sigma f \geq 0 \quad \ldots\ldots\ldots\ldots\ldots\ldots (16)$$

Negative results indicate there is tension on the bottom fiber of the beam. For this condition, successively increasing magnitudes of the prestressing force, F, with the corresponding maximum eccentricity values, e, are substituted until positive results, indicating compression on the bottom fiber of the beam, are obtained. The prestressing force thus obtained is substituted in Eq. 14 and the computed eccentricity is for a zero stress condition.

Losses in Prestress.—The loss in steel prestress is due to creep, shrinkage, and elastic shortening of concrete, creep in steel, and friction.

Elastic Shortening of Concrete.—Since the tendons in a posttensioned member are stressed in succession, the loss in prestress for each tendon will differ. However, for practical purposes, it is sufficiently accurate to assume the same loss for each tendon.

Creep in Concrete.—This loss in prestress is taken as 2.25 times the elastic shortening.

Shrinkage of Concrete.—Recognizing that some shrinkage has taken place prior to prestressing, an average shrinkage strain of 0.0002 is used.

Creep in Steel.—The loss in prestress due to creep in steel is taken as 4% of the stress in the steel immediately after prestressing.

The total loss in steel prestress resulting from creep, shrinkage, and elastic shortening in concrete, and creep in steel is

$$\Delta f_s = 1.0400\, E_s \left(0.0002 + 3.25 \frac{f_{cs}}{E_{ci}}\right) + 0.0400\, f_{se} \quad \ldots\ldots (17)$$

where the flexural modulus of elasticity of concrete at time of prestressing is given as

$$1.8 \times 10^6 + 500\, f'_{ci}$$

The major friction losses in posttensioned members are due to the curvature of the tendons and the friction between the contact materials. The stress at the prestressing jack is

$$T_o = T_x e_n^{(K L + \mu\alpha)} \quad \ldots\ldots\ldots\ldots\ldots\ldots (18)$$

and the prestressing force for a tendon having an area A'_s becomes

$$F'_j = A'_s \left(f_{se} + \Delta f_s \right) e_n^{(K L + \mu \alpha)} \quad \ldots\ldots\ldots\ldots (19)$$

The total angular change α (Fig. 8(a)) is

$$\tan \alpha = \frac{2e}{L/2} = \frac{e}{3 L} \quad \ldots\ldots\ldots\ldots\ldots\ldots (20)$$

where L is in feet and e is in inches.

Finally, the prestressing force at the jack is

$$F'_j = A'_s \left(f_{se} + \Delta f_s \right) e_n^{(KL + \mu e/3L)} \quad \ldots\ldots\ldots\ldots (21)$$

Ultimate Strength.—As an approximation, the ultimate resisting moment of the beam is assumed supplied by a couple T_u j d or C_u j d, where T_u is the ultimate tensile force, C_u is the compressive force supplied by the concrete, and j d is the distance between the two forces (Fig. 9). It is assumed that failure in bond and shear is prevented and that the proper keys and/or shear connectors are provided.

By specifications the ultimate strength capacity of the beam must exceed

$$m_u = DL + 3(LL + I) \quad \ldots\ldots\ldots\ldots\ldots\ldots (22a)$$

or

$$m_u = 2 (DL + LL + I) \quad \ldots\ldots\ldots\ldots\ldots\ldots (22b)$$

whichever is greater.

Shear.—It is assumed that shear failure will not occur before ultimate flexural strength is developed and therefore the principal tensile stress at design loads is not considered. The critical section for shear is taken a distance from the support equal to one and a half times the depth of the member. The difference between the centroid of the beam at midspan and at the supports is small and therefore neglected. The trajectory of the center of gravity of the prestressing steel is assumed parabolic (Fig. 8(b)).

The tangent of the angle β is

$$\tan \beta = \frac{2 e \left(\frac{L}{2} - x \right)^2}{\left(\frac{L}{2} \right)^2} \quad \ldots\ldots\ldots\ldots\ldots\ldots (23)$$

The shear carried by the tendons becomes

$$V_F = A_s f'_s \tan \beta \ldots\ldots\ldots\ldots\ldots\ldots (24)$$

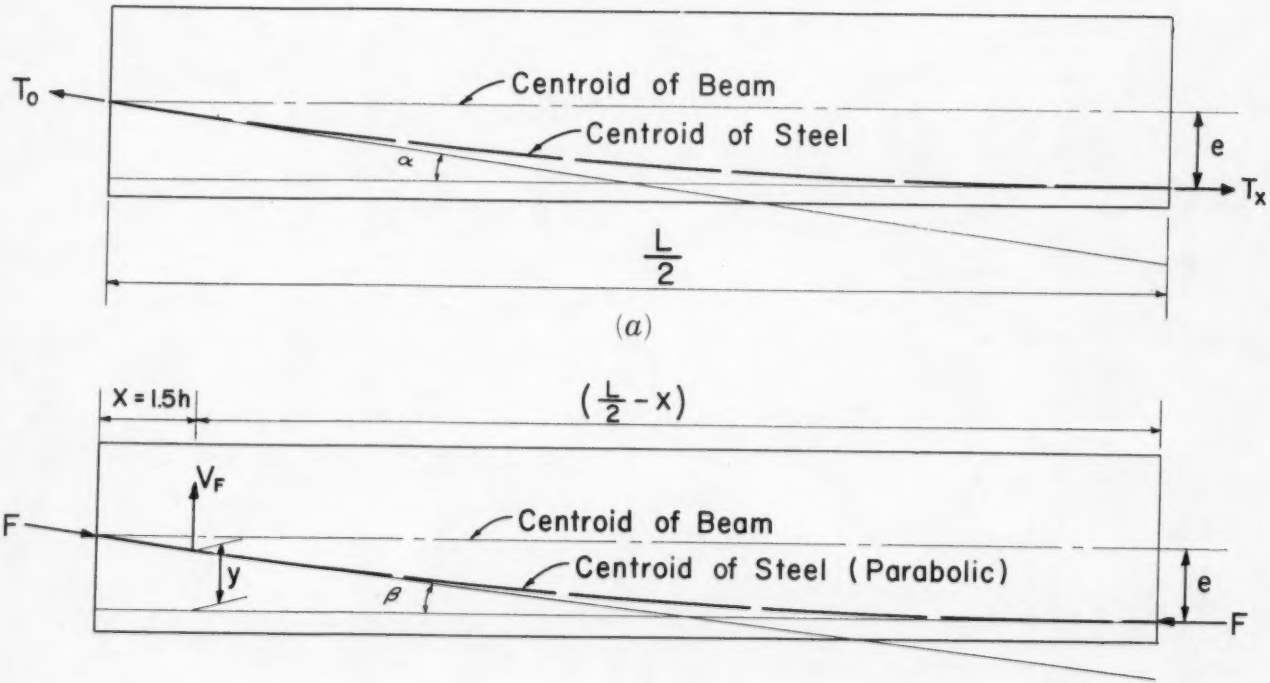

T_0
Centroid of Beam
Centroid of Steel
α
e
T_x
$\frac{L}{2}$
(a)
X = 1.5h
$(\frac{L}{2} - X)$
V_F
F
Centroid of Beam
Centroid of Steel (Parabolic)
y
β
e
F
(b)

FIG. 8

The maximum live load shear for a 10-ft lane is obtained by placing on the span either the standard AASHO H20-S16-44 truck or the lane loading. The standard truck is placed on the span as shown in Fig. 2(b). Letting x equal 1.5 h the maximum live load shear for the truck loading is

$$V = 72 - \frac{1}{L}(9\,h + 672) \quad \ldots\ldots\ldots\ldots\ldots\ldots (25)$$

The shear at a point x due to a uniform load of 0.640 kip per ft and a concentrated load of 26 kips is expressed as

$$V = 0.08\,(4\,L - h) + 32.5\left(\frac{8\,L - h}{L}\right) \quad \ldots\ldots\ldots\ldots (26)$$

By specifications the required ultimate capacity of the beam must exceed

$$V_u = V_D + 3\,V_L - V_F \quad \ldots\ldots\ldots\ldots\ldots (27a)$$

or

$$V_u = 2\left(V_D + V_L\right) - V_F \quad \ldots\ldots\ldots\ldots\ldots (27b)$$

whichever is greater.

The maximum spacing of the web reinforcement equals three-quarters the depth of the member and the minimum area of web reinforcement is

$$A_v = 0.0025\,B'_s \quad \ldots\ldots\ldots\ldots\ldots\ldots (28)$$

The recommended area of web reinforcement is

$$A_v = \frac{1}{2}\,\frac{\left(V_u - V_c\right)}{f'_y\,j\,d}\,s \quad \ldots\ldots\ldots\ldots\ldots (29)$$

where s is the maximum spacing of the web reinforcement. The shear taken by the concrete is

$$V_c = 0.06\,f'_c\,B'\,j\,d \quad \ldots\ldots\ldots\ldots\ldots (30a)$$

or

$$V_c = 180\,B'\,j\,d \quad \ldots\ldots\ldots\ldots\ldots\ldots (30b)$$

whichever is smaller.

USE OF THE COMPUTER

The program developed consists of a main program recorded on a primary tape and input data listed on secondary or supplementary tapes. The pre-

stressed concrete beams are designed using the main program which is independent of the prestressing system used. The secondary tapes are dependent on the prestressing system and supply the required data for the system chosen. The main program is divided into four parts, as shown in the simplified flow chart in Fig. 10.

The input tapes list for each beam the various combinations of prestressing steel area and maximum eccentricity for a given prestressing system. The AASHO-PCI and BPR beams are listed together in order of increasing areas.

The information necessary for the solution of a given problem consists of the following:

1. The area of the interior beam selected from the charts (Figs. 5 and 6).
2. The area of the exterior beam.
3. The dimensions of the structure such as the span length, stringer spacing, slab thickness, and size and number of diaphragms.

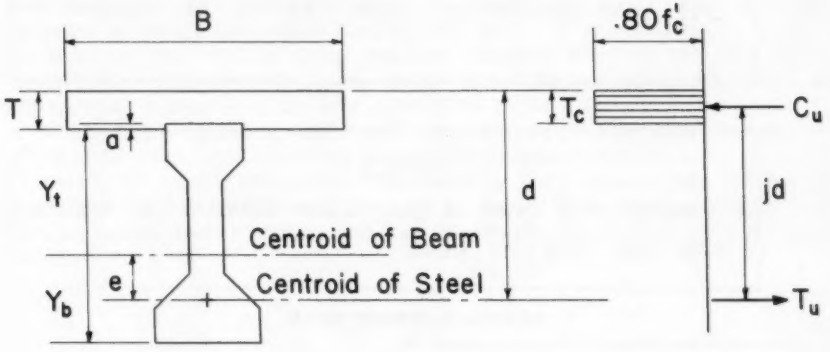


FIG. 9

The combinations of the prestressing steel area and maximum eccentricity for the selected interior beam must be placed in predetermined locations for later use in the program. The computer obtains from the supplementary tape the input for each succeeding beam (Fig. 11(a)) and solves the equation

$$A = A_i \geq 0 \quad \ldots\ldots\ldots\ldots\ldots\ldots (31)$$

Negative results direct the computer to obtain the input for the next beam and zero results indicate the computer has obtained the input for the exterior beam.

Continuing, the computer obtains the flexural modulus of elasticity of the concrete at the time of prestressing and under working loads. The impact fraction, the effective steel prestress, and the maximum live load moment and shear (including impact) for a 10-ft lane are determined. They are computed using the method of decisions based on positive and negative results previously developed.

The weight of the beam and the weight of the concrete slab it supports are computed and are used to determine the design dead load moment and shear for the interior beam. The fraction of a wheel load is applied to the beam to

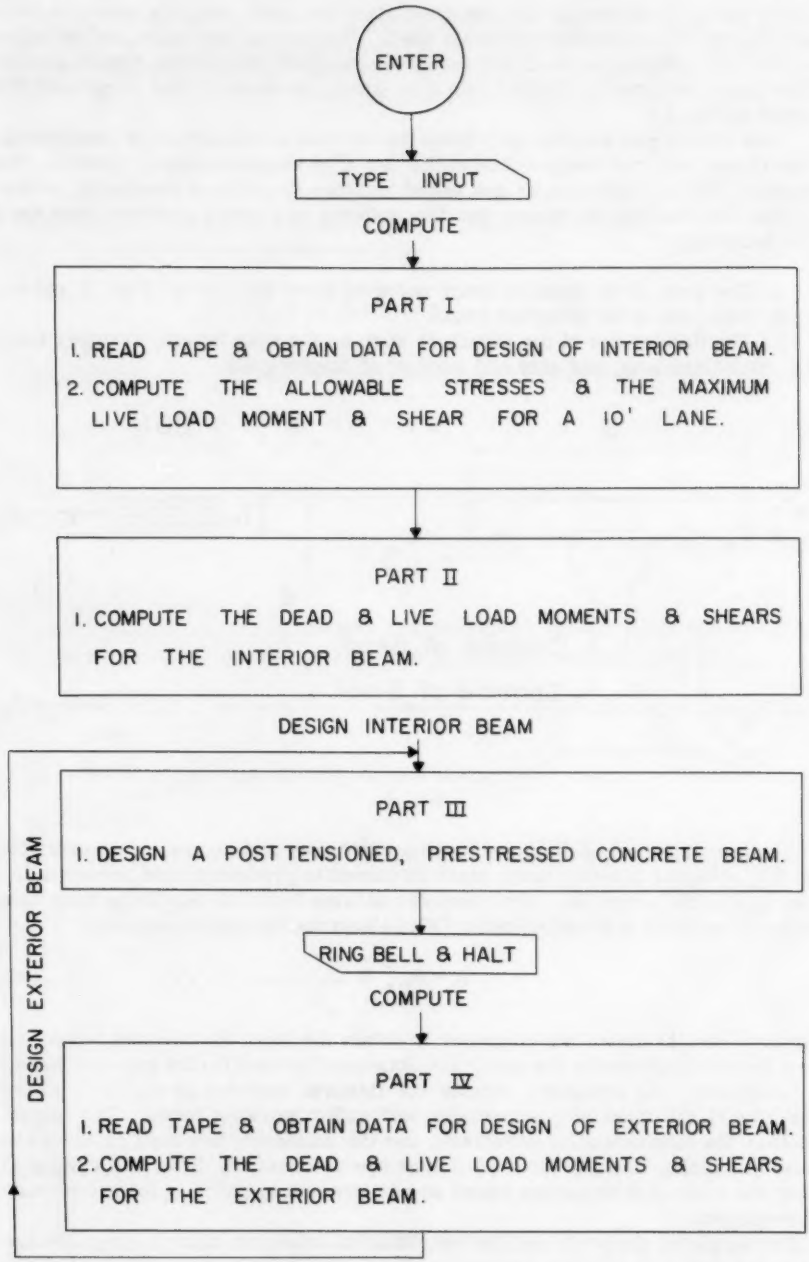

ENTER
TYPE INPUT
COMPUTE
PART I
1. READ TAPE & OBTAIN DATA FOR DESIGN OF INTERIOR BEAM.
2. COMPUTE THE ALLOWABLE STRESSES & THE MAXIMUM LIVE LOAD MOMENT & SHEAR FOR A 10' LANE.
PART II
1. COMPUTE THE DEAD & LIVE LOAD MOMENTS & SHEARS FOR THE INTERIOR BEAM.
DESIGN INTERIOR BEAM
PART III
1. DESIGN A POST TENSIONED, PRESTRESSED CONCRETE BEAM.
RING BELL & HALT
COMPUTE
PART IV
1. READ TAPE & OBTAIN DATA FOR DESIGN OF EXTERIOR BEAM.
2. COMPUTE THE DEAD & LIVE LOAD MOMENTS & SHEARS FOR THE EXTERIOR BEAM.
DESIGN EXTERIOR BEAM

FIG. 10.—FLOW CHART

determine the design live load bending moment and shear. The computer is now ready to design the interior beam.

The effective flange, the composite section properties and the stresses on the top and bottom fibers of the beam due to external loads are computed. An initial prestressing steel area is selected from the previously obtained input and the prestressing force is then computed using Eq. 15. The substitution of the prestressing force, F, and its corresponding maximum eccentricity, e, in Eq. 13 as previously explained, will indicate the state of stress on the bottom fiber of the beam. The computer will select successive areas of steel until the desired positive results are obtained. If all possible combinations of steel area and maximum eccentricity fail to provide satisfactory results, the computer will halt. The required eccentricity for zero stress condition is computed using Eq. 14.

The area of the beam, the area of the prestressing steel and the required eccentricity are then typed by the computer (Fig. 12). The losses in prestress are determined in order to compute the initial prestressing force; the prestressing force required at the jacks for the separate tendons and the initial stresses on the top and bottom fibers of the beam. The prestressing force at the jacks and the resultant initial and final concrete stresses are then typed. The ultimate strength of the beam, the shear and the deflections are computed and typed as indicated on the flow chart (Fig. 11(b)).

The halt is provided so that the critical stresses and the ultimate strength of the beam can be checked. It is possible to compute again using the next area of steel if the results found are unsatisfactory. The computer then obtains the input data for the exterior beam using the process previously described for the interior beam, and by returning to Part III, the computer designs the exterior beam.

CONCLUSIONS

The programs described herein select and design the standard AASHO-PCI and BPR posttensioned prestressed concrete beams with AASHO H20-S16-44 loading. The flexibility of the programs provide for the selection and design of any shape beam using any AASHO live loading and strength of concrete and steel.

The programs presented are applicable to building construction if they are modified as follows. First, the live load is different since it usually consists of a uniformly distributed load instead of the standard AASHO loadings. Second, although the shapes of the beams differ from the heavy bridge beams this consideration is of minor importance since only the physical properties of the beam are required in the programs. The third consideration is the effect of concentrated loads on the profile of the prestressing tendons. A parabolic curve is still possible since the envelope of the moment diagram of a unit load moving across the span is a parabola. Using the AASHO loadings, it is the experience of the writer that because of the parabolic shape of the cables and the moment envelope, the stresses at points other than midspan are not critical and therefore they are not considered.

The AASHO Specifications limit the live load deflection to 1/800 of the span for steel beams and girders. If this criteria is allowed to govern for prestressed concrete beams it then becomes unnecessary to calculate live load deflections for two reasons:

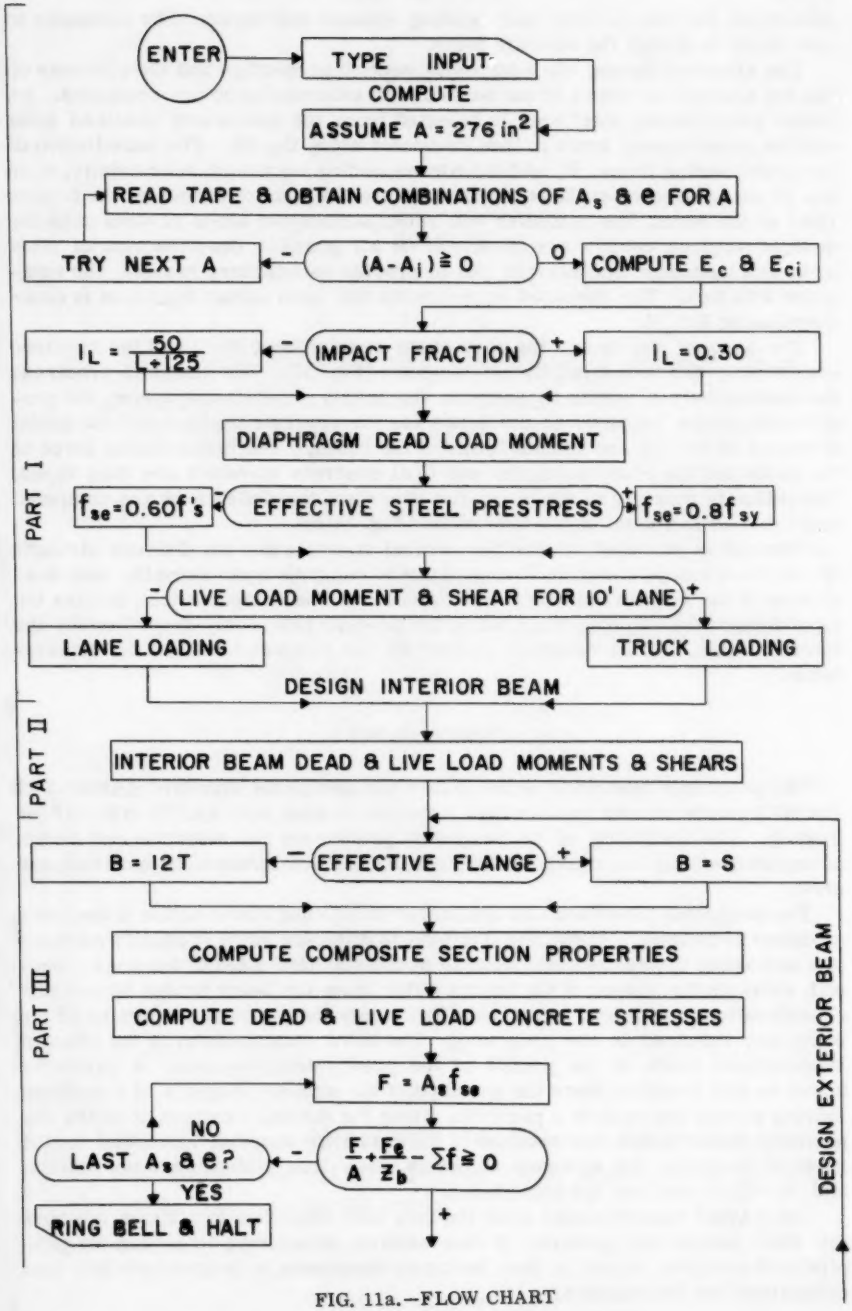


FIG. 11a.—FLOW CHART

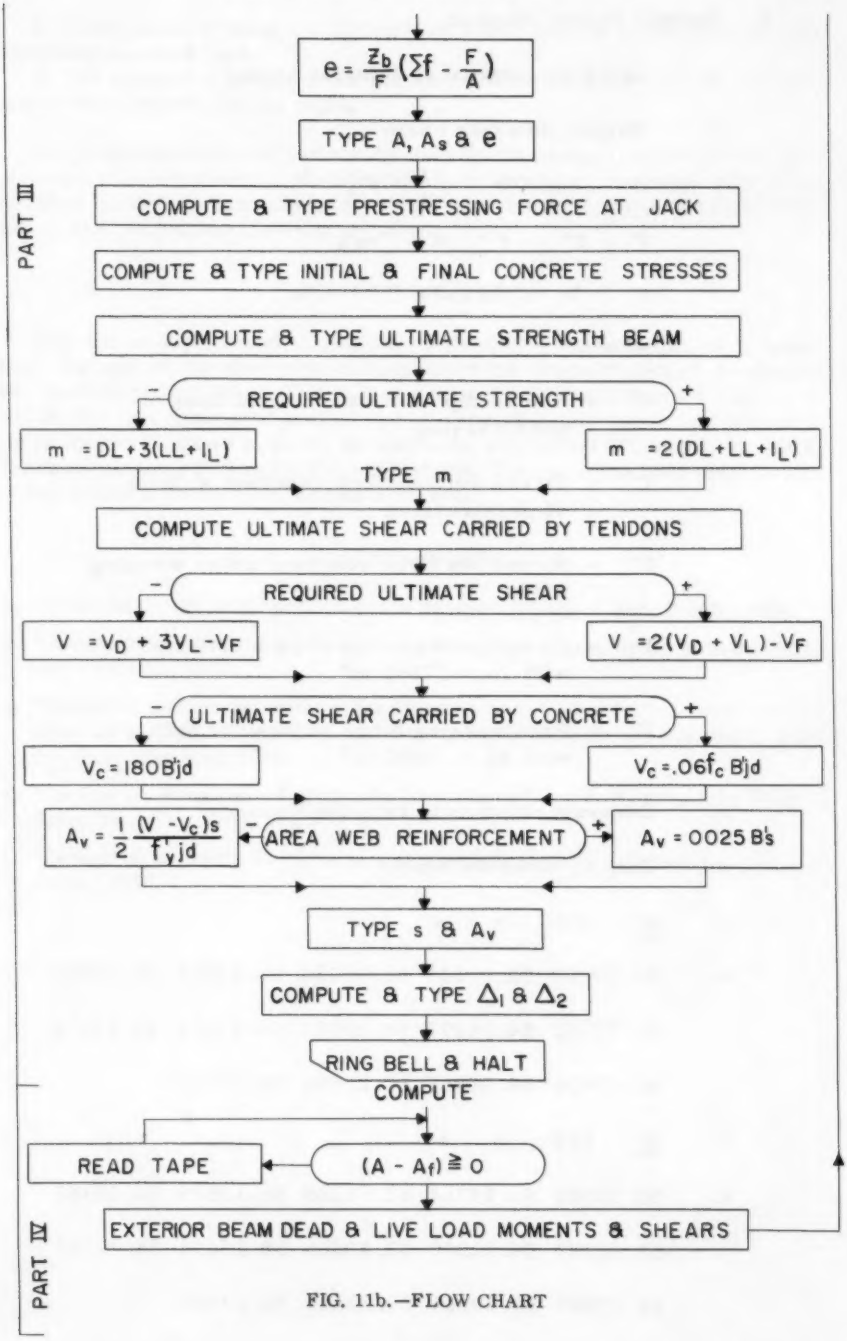

$e = \frac{Z_b}{F}(\Sigma f - \frac{F}{A})$
TYPE A, A_s & e
PART III
COMPUTE & TYPE PRESTRESSING FORCE AT JACK
COMPUTE & TYPE INITIAL & FINAL CONCRETE STRESSES
COMPUTE & TYPE ULTIMATE STRENGTH BEAM
REQUIRED ULTIMATE STRENGTH
m = DL+3(LL+I_L)
m = 2(DL+LL+I_L)
TYPE m
COMPUTE ULTIMATE SHEAR CARRIED BY TENDONS
REQUIRED ULTIMATE SHEAR
V = V_D + 3V_L - V_F
V = 2(V_D + V_L) - V_F
ULTIMATE SHEAR CARRIED BY CONCRETE
V_c = 180B'jd
V_c = .06f'_c B'jd
$A_v = \frac{1}{2}\frac{(V - V_c)s}{f'_y jd}$
AREA WEB REINFORCEMENT
A_v = 0025 B's
TYPE s & A_v
COMPUTE & TYPE Δ1 & Δ2
RING BELL & HALT
COMPUTE
READ TAPE
(A - A_f) ≅ 0
PART IV
EXTERIOR BEAM DEAD & LIVE LOAD MOMENTS & SHEARS

FIG. 11b.—FLOW CHART

1. Sample Output Format

   a. Compute command, interior beam

   b. Output, interior beam

   $A \quad A_s \quad e \quad F_j' \quad F_j''$

   $f' \quad f'' \quad f''' \quad M_u \quad m_u$

   $s \quad A_v \quad \Delta_1 \quad \Delta_2$

   where

   f' = Stress top fiber concrete at time of prestressing

   f'' = Stress bottom fiber concrete at time of prestressing

   f''' = Stress top fiber concrete under working loads

   F_j' = Prestressing force at jack for unit with $A_s' = .7856$ in²

   F_j'' = Prestressing force at jack for unit with $A_s' = .5892$ in²

   c. Compute command, exterior beam

   d. Output, exterior beam

2. a. qf 000 z s

   b. 53.36900 51.21604 51.90766 56.13867 56.10400

   52.79632 54.16397 54.10901 58.17585 58.17278

   52.27000 50.88865 50.28555 50.17101

   c. qf 610 z s

   d. 53.36900 51.15712 52.11228 56.13575 56.10181

   -52.42389 54.12399 53.80865 58.13654 58.11707

   52.27000 50.40500 50.23685 50.14581

FIG. 12

1. There usually results a net upward deflection due to the effect of prestressing and dead load.
2. The moment of inertia of the standard beams are quite large in comparison to steel beams of equal depth.

The programs presented can also be used for the design of pretensioned prestressed concrete beams. Modifications are necessary for bond, straight or deviated (with hold-down devices) cables, concrete stresses at the ends of the beam, and the greater losses in prestress.

ACKNOWLEDGMENTS

The writer is indebted to Lockwood, Kessler & Bartlett, Inc., of Syosset, N.Y. for use of the electronic computer and the responsibility of developing the described programs. The writer gratefully thanks Edward J. Squire, F. ASCE, for his assistance and advice in preparing this paper and wishes to thank Joseph M. Dawson, Ennio M. Gherardi, and Robert O'Connell, M. ASCE, for their valuable comments and suggestions. The paper has been used as part of the writer's thesis for a Master's degree.

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1. "Criteria for Prestressed Concrete Bridges," Bur. of Pub. Roads, 1954.
2. "Standard Specifications for Highway Bridges," Amer. Assn. of State Highway Officials, 1957.
3. "Tentative Recommendations for Prestressed Concrete," Report on the Joint ACI-ASCE Committee on Prestressed Reinforced Concrete, Proceedings, January, 1958.
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5. "Design of Highway Bridges in Prestressed Concrete," Portland Cement Assn., 1958.

Journal of the
STRUCTURAL DIVISION
Proceedings of the American Society of Civil Engineers

DIGEST OF THE GUIDE TO DESIGN CRITERIA FOR METAL COMPRESSION MEMBERS

By Bruce G. Johnston,[1] F. ASCE

FOREWORD

The papers that form the ASCE-CRC Symposium on Metal Compression Members are representative of all phases of Column Research Council activities. Several papers represent summaries of both past and current research, prepared specifically to assist in the preparation of the "Guide to Design Criteria for Metal Compression Members." Also included are reports on recently completed research on pony truss bridges and on plate girders.

In the January 1960 Journal of the Structural Division Bruce G. Johnston documented the inception of the CRC and the many activities undertaken by the Council.

Thomas C. Kavanagh, in the February 1960 Journal of the Structural Division, reviewed the state of knowledge concerning the effective length of framed columns.

In the April Journal of the Structural Division, Robert M. Barnoff and William G. Mooney studied the effect of floor-system participation on pony truss bridges.

SYNOPSIS

Eleven "highlights" from the forthcoming CRC "Guide to Design Criteria for Metal Compression Members" are presented. These items are, in general,

Note.—Discussion open until September 1, 1960. Separate Discussions should be submitted for the individual papers in this symposium. To extend the closing date one month, a written request must be filed with the Executive Secretary, ASCE. This paper is part of the copyrighted Journal of the Structural Division, Proceedings of the American Society of Civil Engineers, Vol. 86, No. ST 4, April, 1960.

[1] Chmn. of Column Research Council and ASCE Committee on Compression Members, Prof. of Struct. Engrg., Civil Engrg. Dept., Univ. of Michigan, Ann Arbor, Mich.

the simpler and less technical aspects of the Guide. They were chosen because they represent contributions to the current status of metal compression member design.

INTRODUCTION

This paper provides a prepublication preview of the Column Research Council "Guide to Design Criteria for Metal Compression Members" that is scheduled to be published in 1960. Hereinafter, this forthcoming publication will be referred to simply as the "Guide." Rather than attempt to produce a uniform reduction of a work that has already been condensed, the writer will list eleven significant contributions of the Guide to the current (1960) status of metal compression member design. In general, the simpler and less technical aspects of the Guide will be emphasized in this digest. Of course, the real credit for these contributions goes directly to the many individuals who have carried out Column Research Council (CRC) projects, or who have prepared summaries of existing information in a particular area.

These "highlights" of the Guide, as they might be termed, are drawn as follows from the chapters of the Guide:

Two from Chapter 2 on "Centrally Loaded Columns";
Two from Chapter 3 on "Compression Member Details";
Four from Chapter 4 on "Laterally Unsupported Beams"; and
Three from Chapter 5 on "Beam-Columns."

A number of previous papers of this symposium were prepared specifically for the Guide and in some cases the discussion of a guide highlight will be shortened to avoid duplication with another symposium paper.

Although the writer of this digest has acted as editor of the Guide, the Guide itself represents, in a very substantial sense, the combined effort of CRC as a whole. All members of the CRC have had the opportunity of reviewing prepublication drafts, and many detailed suggestions by various members have been incorporated into the text. Moreover, many basic portions of the original text of the Guide have been prepared directly by CRC members who, therefore, are contributing authors of the Guide acting on behalf of CRC. These direct contributions have come in most part from the following:

Austin, W. J.
Beedle, L. S.
Clark, J. W.
Higgins, T. R.
Hill, H. N.
Jombock, J. R.
Jones, Jonathan
Julian, O. G.
Kavanagh, T. C.
Ketter, R. L.
Thurlimann, Bruno
Winter, George

The Guide is dedicated to the memory of Shortridge Hardesty, whose influence as chairman and director of CRC, from its very beginning, was a personal inspiration to all who worked with him. Although the Guide itself has been completed in draft form, it has not as yet been "finalized" and various additional revisions may be expected. Thus, the writer must take responsibility for any views expressed herein that may be amplified or modified in the Guide as finally published.

HIGHLIGHTS OF THE GUIDE

1. Whereas the basic column strength of steel and aluminum alloy columns has, in the past, been based generally on two different and contradictory concepts, it is a goal of CRC to unify the evaluation of basic column strength under the tangent modulus strength criterion.

Column buckling theory was initiated by Euler (1),[2] who more than 200 years ago provided the simple formula that bears his name. Euler's formula is directly applicable to design in the long column range. Modifications in this formula that were introduced to adapt it to shorter columns that fail in the inelastic range resulted in two different theories, known respectively as the "tangent modulus theory" (2) and the "double modulus theory" (3). Experimental evidence favored the first named theory, but it was not until 1947 that F. R. Shanley (4) provided the key to the significant interrelation between the two theories. Although the Euler and the associated tangent modulus formulas found direct application in the airplane industry in the design of aluminum alloy struts, similar application to heavy steel construction lagged because of the discrepancy between tangent modulus strength predictions, based on small coupon stress-strain curves, and the actual strengths of tested columns. As is well known, small coupon tests of steel usually show practically linear stress-strain relationship up to loads very near the yield point. In steel columns, residual stresses approaching half the yield point are "locked" in the member, as a result of uneven cooling of hot-rolled steel on the mill cooling beds. Although the residual stress levels are probably greater in rolled WF or I members, they are present in angles, universal mill plate, and other elements of built-up columns.

It became apparent in the late 1940's that the key to the application of the tangent modulus concept to the steel column lay in the inclusion of the residual stress effect on the average stress-strain diagram of the entire steel member. As a result of this recognition of the importance of the residual stress effect on column behavior, CRC, through its research committees, initiated work on the influence of this variable. It was obvious that both theoretical and experimental investigations would be required to prove the validity of the concept, and a considerable number of studies and projects have resulted, primarily at Lehigh University (5,6,7,8), some of which have been wholly or partly sponsored by the CRC. Although much investigational work remained, it became possible in May of 1952 for CRC to release the following statement in its Technical Memorandum No. 1 on "The Basic Column Formula" (9):

> "It is the considered opinion of the Column Research Council that the tangent modulus formula affords a proper basis for the establishment of working load formulas."

The reason for emphasis on this statement lies in the fact that it is not restricted to any particular metal. The key to the application of the tangent modulus concept to steel columns is the determination of the tangent modulus of the effective stress-strain diagram of the entire cross section. The principal factors that cause differences between the effective stress-strain curve of the complete cross section and the individual stress-strain curve of a test coupon are the presence of residual stress and the variation in yield point over the cross section. The failure of the tangent modulus concept (when based on individual coupon tests) to explain the behavior of steel column tests in the past

[2] Numerals in parentheses, thus (1), refer to corresponding items in the Bibliography.

has usually been attributed to the presence of accidental end-eccentricities. For any given test, by means of the secant formula, a fictitious end-eccentricity can always be determined that will exactly account for the column strength on the assumption that the column fails when the maximum stress due to combined direct load and bending reaches the yield point, the material being assumed as perfectly elastic up to the yield point. Although accidental end-eccentricity does play a part in determining the strength of a column that is intended to be centrally loaded, it is now known to be secondary to the role of residual stress. Thus, the effect of a major factor (residual stress) hitherto has been effectively included in the design application of test results by means of the empirical magnification of what is frequently a minor factor (accidental eccentricity). The secant formula approach also lacks generality since it is not applicable at all to non-ferrous alloys for which there is no well defined yield point.

In their symposium paper, prepared for CRC, L. S. Beedle and Lambert Tall (10) review the development of the basic theory by means of which the residual stress influence on column strength is evaluated.

Since the departure from linearity in the effective stress-strain curve for a steel column is explained by residual stress, the effective proportional limit may be replaced by

$$\sigma_p = \sigma_y - \sigma_{RC}$$

and, as pointed out in a special report to CRC (11), the following result is obtained:

$$\sigma_c = \sigma_y - \frac{\sigma_{RC}}{\pi^2 E}(\sigma_y - \sigma_{RC})\left(\frac{KL}{r}\right)^2 \quad \ldots\ldots\ldots\ldots \quad (1)$$

If σ_{RC} is taken as one half the yield point (about the maximum usually observed in wide flange shapes), Eq. 1 becomes

$$\sigma_c = \sigma_y - \frac{\sigma_y^2}{4\pi^2 E}\left(\frac{KL}{r}\right)^2 \quad \ldots\ldots\ldots\ldots \quad (2)$$

Eq. 2 merges with the Euler curve at a point of common tangency and is here recommended for the establishment of basic column strength curves for constructional steels having a yield point comparable to A7 steel. It is conservative for higher strength steels. It is interesting that Eq. 2 can be reduced to be identical to Eq. 1.395 of ANC-5, Strength of Metal Aircraft Elements, March, 1955. Also George Winter (12) presented the identical equation in 1946 and stated that it gave "a very close approximation of the values of the secant formula in the low and medium range of L/r." Thus, while the formula herein has been related to the effects of residual stress in wide flange shapes, the identical formula has been shown to be consistent with the secant formula which incorporates an end-eccentricity ratio determined so as to make the formula fit a large body of column test results.

In a survey prepared for CRC by Beedle and Huber (13), it was recommended that the minimum ASTM Specification yield point in tension be adopted as a suitable value for σ_y in compression as used in Eq. 2. Although overconservative, this recommendation is tentatively extended to all structural high strength steels for which ASTM Specifications are provided, together with a similar extension of the assumption that $\sigma_{RC} = \sigma_y/2$, effectively, for introduction into Eq. 2.

In Fig. 1 are plotted the actual critical average stresses at the tangent modulus load (as predicted by Eq. 2) for all structural steels for which there are currently available ASTM Specifications. The stresses for the two aluminum alloys are determined directly from the tangent modulus curves as given in the ASCE Specifications (14,15).

In the Guide, the data of Fig. 1 are presented in tabular form. The Guide then proceeds to discuss the difficult matter of factor of safety, by which the column strength curves of Fig. 1 are related to column design curves or formulas at allowable stress levels. Various existing design formulas are discussed briefly.

2. The design of compression members should include a proper consideration of restraint, or lack of restraint, at the ends.

The importance of proper consideration of "effective length" in column design specifications has been well expressed by T. C. Kavanagh (16):

> "Elementary theory teaches that a pin-ended column has an effective length equal to its actual length, but such idealization is not necessarily applicable in practice because of the varied end restraints present. The dilemma may be most readily recognized in the light of the correct statement that, theoretically at least, it is possible for the effective length of a column to vary anywhere from zero to infinity times the actual column length. It is a matter of awe that a subject having so fundamental an effect on both the safety and economy of the structure has been so blandly ignored in specifications in the past."

Reference in this section is made exclusively to centrally loaded columns for which intentional lateral loads and eccentricities of axial load are absent. T. C. Kavanagh has also contributed to this symposium a general review of this subject (17), much of which will be found in condensed form in the Guide. The Guide will also include nomograms to provide for the rapid calculation of the effective length of framed columns. Since these are available in Mr. Kavanagh's symposium paper (February 1960), they will not be repeated herein.

It is convenient in design to break down the structure into plane frameworks and to consider the flexural buckling of each member in the plane of the main framework of which it is a part (such as a main bridge truss), and again as buckling normal to the main framework (sometimes spoken of as "out of the plane") such as that which takes place in the plane of the horizontal bracing of a bridge truss.

The compression member in a plane framework may be characterized as to the type of possible buckling failure as follows:

(a) That in which the translational restraint at the ends of the member is so great that little or no translation takes place, such as in truss frameworks.

(b) That in which both translation and rotation of the ends of the member are possible, such as is usually found in open continuous frames.

(c) That in which rotational stiffness at the ends of the member is so great that little or no rotation takes place, as in a rectangular portal frame with extremely heavy or stiff upper horizontal cross member.

Special cases of the above would include the condition wherein either the translational or rotational rigidity (or both) at one (or both) ends of the compression member is zero. (One example would be a rectangular continuous frame with vertical columns fixed at their bases but hinged at their tops.)

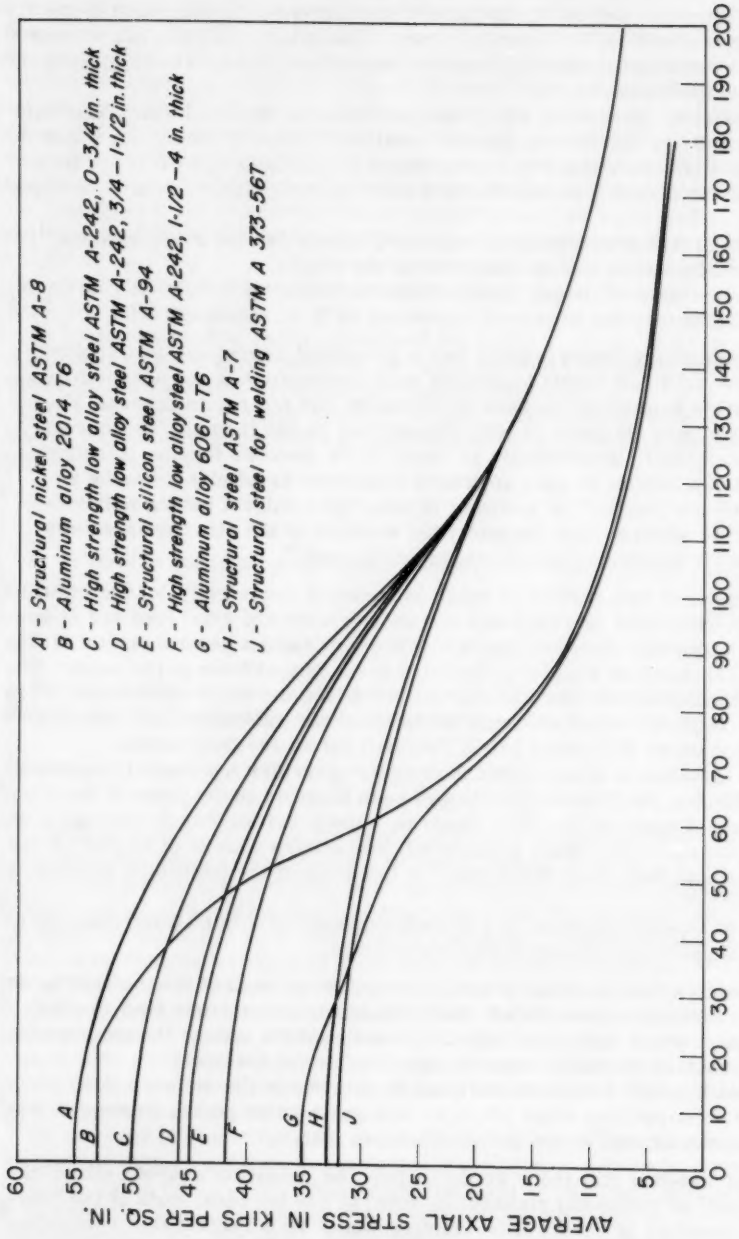
A - Structural nickel steel ASTM A-8
B - Aluminum alloy 2014 T6
C - High strength low alloy steel ASTM A-242, 0-3/4 in. thick
D - High strength low alloy steel ASTM A-242, 3/4 - 1-1/2 in. thick
E - Structural silicon steel ASTM A-94
F - High strength low alloy steel ASTM A-242, 1-1/2 - 4 in. thick
G - Aluminum alloy 6061 - T6
H - Structural steel ASTM A-7
J - Structural steel for welding ASTM A 373-56T
AVERAGE AXIAL STRESS IN KIPS PER SQ. IN.
KL/r
0 5 10 15 20 25 30 35 40 45 50 55 60
0 10 20 30 40 50 60 70 80 90 100 110 120 130 140 150 160 170 180 190 200

FIG. 1.—BASIC COLUMN STRENGTH IN KIPS PER SQUARE INCH FOR STRUCTURAL METALS.

Fig. 2 provides a table of limiting K values for use in Eq. 2. These values, used with engineering judgment, provide a minimum basis for the inclusion of the column end restraint effect.

These K-values are for idealized conditions in which the rotation and/or translation restraints at the ends of the column are either fully realized or non-existent; in the latter case the strength calculation of the column is independent of adjacent members. At the base, shown fixed under conditions a, b, c, and d in Fig. 2, the condition of full fixity can be approached only when the column is anchored securely by adequate embedment or by a moment-resisting bolted connection to a footing that is designed to resist overturning moment and for which the rotation is negligible. Column condition (a) is approached when the

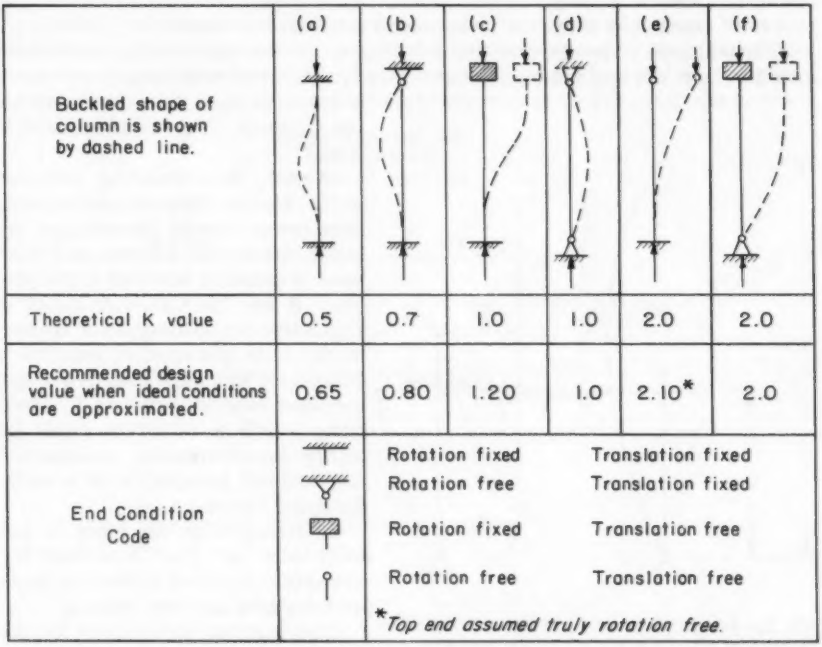

FIG. 2.—EQUIVALENT LENGTHS OF CENTRALLY LOADED COLUMNS WITH VARIOUS IDEALIZED END CONDITIONS.

top of the column is integrally framed to a heavy girder many times more rigid than the column under design. Column condition (c) is the same as (a) except that the translation restraint is either absent or of uncertain amount at the top. Condition (f) is the same at the top as (c) but the condition at the bottom is uncertain as to rotation restraint. The recommended design values of K are a modification of the ideal values, taking account of the fact that perfect fixity is impossible of complete attainment, and, likewise, that a rotation free condition is not attained in practice.

For carbon structural steel columns with slenderness ratios less than about 60, the critical buckling load for a fixed end column (Type (a), Fig. 2) is less

than 10% different from that with pinned ends (assuming no translation). Thus, the introduction of K-values has little if any practical effect on the design of relatively short columns. In the case of low alloy and silicon steels the figure of 60 may be changed to 50 for the same differential of 10%. For aluminum alloy columns the figure varies from about 40 for the higher strength alloys down to 30 for the lower strengths. The difference in buckling loads in the pinned and fixed end conditions increases steadily as the slenderness ratio rises. The type (a) fixed-end column load may be 100% greater than that of the pin-ended column at L/r of about 100 for steel columns and 70 for aluminum alloy construction. Thus the use of K-values here results in appreciable economy for large slenderness ratios. The increasing sensitivity to end restraint as L/r increases, and possible changes in restraint that may result during the life of the structure, suggest the desirability of giving careful attention to end restraint conditions when the members are relatively slender.

If translation of joints, (that is, sidesway or lateral movement) is possible, as in the case of rigid frames, the effective length of columns may be considerably larger than the actual length of the columns. This is illustrated in Fig. 3.

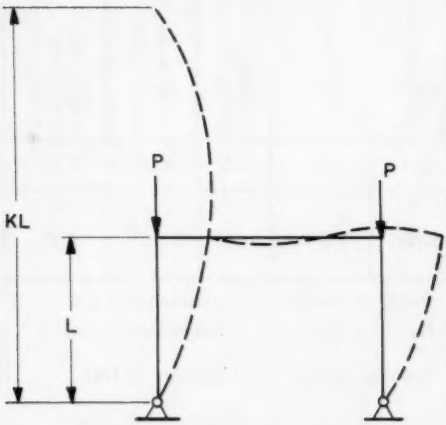

FIG. 3.—FRAME WITH EFFECTIVE COLUMN LENGTH FACTOR K GREATER THAN 2.

Where free-standing columns occur, such as those supporting outdoor crane runway girders and similar structures, a K-value of 2 has been previously noted as appropriate. It has been further noted in Fig. 3 that for some types of frameworks it is possible to imagine a column for which the effective length becomes even larger than 2. Obviously, such a situation would be highly impracticable, uneconomical, and not acceptable in a well-designed structure.

3. Recognition is given to the utilization of the post-buckling strength of flat steel plates that form portions of a built-up column.

Conventional procedures for the determination of permissible width-thickness ratios in heavy steel structures are based on critical buckling stresses determined by classical small deflection theory. The advent of light gage steel construction as well as the use of lighter metals has focussed increasing attention on the utilization of post-buckling strength. Although the Guide does not ignore the conventional procedures, J. R. Jombock and J. W. Clark (18), have provided a survey and evaluation of procedures applicable to thin-gage materials. Note also the paper by the same authors (19) as presented at this symposium.

As evidenced by the symposium paper by K. Basler and B. Thurlimann (20), the extension into the heavier construction field of these design techniques, based on post-buckling strength and applied to plate girders as well as columns, would appear to be imminent.

The economic use of material in any situation requires that permissible stresses be as large as possible. Since the permissible stress in a column decreases with increasing K L/r, it is obvious that the K L/r should be kept as small as possible. The length of the column is governed by geometric arrangement of the structure and the radius of gyration should therefore be as large as possible. In the case of moderately heavy loads this is accomplished by spreading the material in the cross-section and, where necessary, joining this by lacing bars or utilizing perforated cover plates. For very light loads, however, this is uneconomical and the use of very thin material with only part of the column cross section considered effective may be desirable. This leads to the "effective-width" concept that utilizes the "post-buckling" strength of the plate. This approach to design bears a relationship to current specification practice in heavy steel construction that is not unlike the relationship between the plastic limit design and the "working stress" concepts of design. This procedure is applicable not only to long columns carrying small loads but also to cases where columns serve the dual function of providing walls, partitions, bulkheads, or wing or hull "stressed-skin" structures as in aeroplanes or ships.

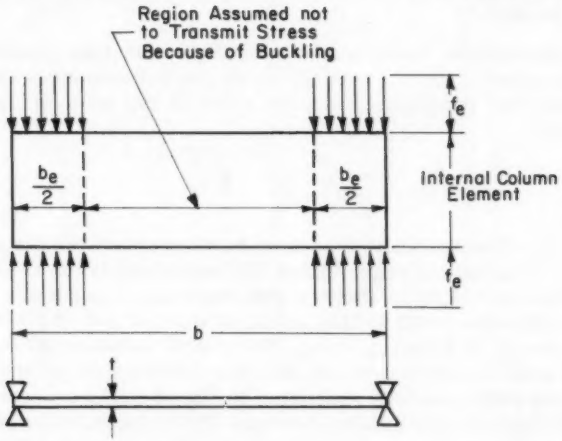

FIG. 4

A flat plate loaded in edge compression and supported along both unloaded edges will develop additional load after the buckling stress for the plate has been exceeded, and it is advantageous in design to take account of the post-buckling ultimate strength. This can be done by the effective width method, in which only a part of the width of the sheet is considered to be effective in carrying load after the local buckling stress of the plate has been exceeded (Fig. 4).

The effective width concept is currently used in at least four domestic specifications. The 1956 edition of the "Specification for the Design of Light Gage Cold-Formed Steel Structural Members" (21) makes extensive use of effective width in the design of light gage steel structures. The specifications of the ASCE Committee of the Structural Division on Design in Lightweight Structural Alloys for structures of aluminum alloys 6061-T6 (14) and 2014-T6 (15) also make use of effective widths. The AISC 1949 "Specification for the Design,

Fabrication, and Erection of Structural Steel for Buildings" (22) tacitly allows the use of an effective width in the design of compression members having plate elements with widths greater than 40 times the thickness. Industrial buildings also provide useful applications of stiffened sheet construction. In this area of application, a very readable and easily understood discussion of the effective width concept as applied to light gage steel design has been prepared by Winter (23).

The first to use the effective width concept to develop expressions for the ultimate strength of plates was von Karman (24). Many other formulas for the effective width have been derived, some empirical, some based on approximate analysis, and some based on large-deflection theory with varying amounts of rigor. Jombock and Clark (18) list fourteen effective width formulas along with their sources and give remarks regarding assumptions upon which they are based. They suggest that the results of these comparisons may be summed up as Gerard did in Reference (25) page 522:

> "Of all the theories shown, Equation (3 herein) apparently gives the best fit to the test data and is still predominantly conservative throughout the range of the data."

For ease of application, there seems to be little advantage of any of the other formulas. As given by von Karman (24) as an approximate expression based on the assumption that two strips along the sides on the verge of buckling carry the entire load:

$$b_e = \left[\frac{\pi}{\sqrt{3(1-\nu^2)}} \sqrt{\frac{E}{f_e}} \right] t \quad \ldots\ldots\ldots\ldots \quad (3)$$

Although the effective width also may be determined for plates supported along only one longitudinal edge, Winter (23) concludes, in his recent commentary on the AISI Specification, that the post-buckling strength of a plate in this condition provides too small a differential to warrant use of other than critical buckling stress as a design basis. The practical consequence in thin gage cold-formed metal construction has been the development of shapes in which outer edges are quite generally supported by lipped stiffeners, cold-formed by bending the main plate along its outer edge. Thus, the unstiffened plate is not usually encountered in practical applications of thin gauge metals.

Since the effective width concept has been well developed in current specifications and commentaries (14,15,21,23), reference is made thereto for further information on the subject.

4. Recommendations are made for the design of bridge compression members with perforated cover plates.

In the design of the Huey Long cantilever bridge over the Mississippi River just above New Orleans, La., in the early 1930's, the firm of Modjeski and Masters replaced the then conventional plane or planes of lacing in truss compression members with cover plates containing axially spaced holes of substantial size to afford access for riveting and painting. Whereas the lacing members previously used could not be counted upon to resist the axial compression, the portions of these perforated cover plates outside the perforations were included in member area. In addition to this economy, additional advantages arose from the reduction of fabrication and maintenance costs of such members, and their superior stiffness and straightness. The use of perforated cover plates has since become general, although design practice in their use has varied.

The foregoing paragraph (with minor amplification) is taken from a review and design study of perforated cover plates carried out under the sponsorship of the American Association of Railroads and Modjeski and Masters at Lehigh University by White and Thurlimann (26), with CRC acting in an advisory capacity. This report provides a digest of earlier investigations at the National Bureau of Standards (27), and gives specific recommendations for design of columns with perforated cover plates. Rules for proportioning cover plate details are also given in the latest AASHO Specifications for Highway Bridge Design (28), (7th ed.) Sec. 1.6.36(b).

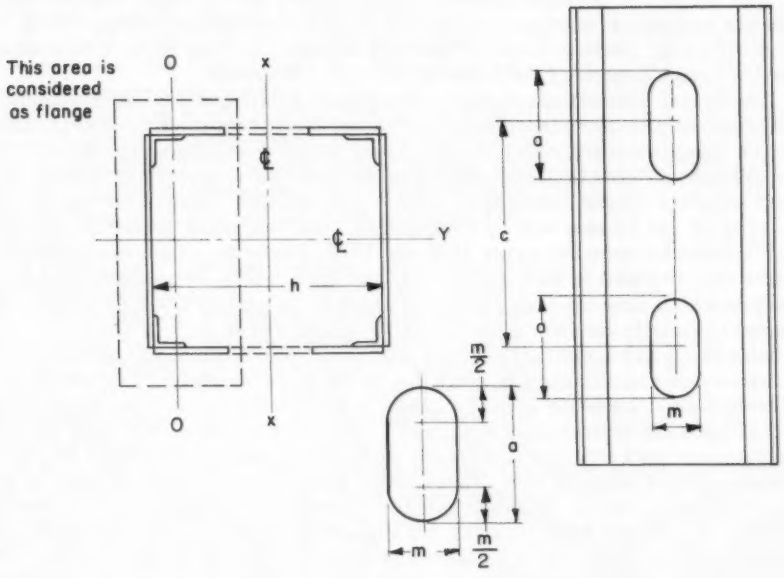

FIG. 5.—PERFORATED PLATES IN COMPRESSION CHORDS

The following design suggestions are to some extent a simplified version of some of the White-Thurlimann recommendations combined with some features of the current AASHO recommendations:

(1) The shape of the perforations should be ovaloid (that is, straight sides with semi-circular ends). elliptical or circular. For the first two cases, the long axis should be in the direction of the column axis.

(2) The clear distance (c-a) (Fig. 5) between perforations should not be less than the unsupported distance between the nearest lines of connecting rivets.

(3) The net section of the column (defined as the gross cross section less the section of the perforations) should be used in computing the axial rigidity and the section moments of inertia about the x and y axes.

(4) If the slenderness ratio a/r_o (length of perforation/radius of gyration of flange) is equal to or less than 1/3 of L/r_x with a maximum of 20, the code values for allowable column stress, applied to the net cross-sectional area, may be used directly to determine the permissible load.

(5) At the point of perforation, each flange should be designed to resist half the total transverse shear force. Checking of the transverse shearing stress in members with at least one solid transverse web plate is not necessary. Perforated cover-plates designed in accordance with rule 2 need not be checked for longitudinal shear.

(6) The transverse distance from the perforation to the nearest line of connecting rivets or point of support, divided by the plate thickness, shall conform to the specification requirements for plate thickness in main compression members.

The subject of perforated cover plates is only one of many similar topics that are included in the Guide in Chapter 3 on "Compression Member Details."

5. If a long shallow beam is laterally unsupported, the Guide recommends careful consideration of the possible use of a box girder.

The initial theoretical solution of the elastic buckling of a beam of rectangular cross-section was presented by L. Prandtl (29) in 1899 for a number of cases, and an independent and nearly simultaneous solution was made by Michell (30) for the case of the simply supported beam under pure (constant) bending moment supplied by end moment couples. The earliest solution for the lateral buckling of the I-beam was by Timoshenko, who published papers in Russian and German between the years 1906 and 1910. These and other early developments are reviewed by Timoshenko (31) and Bleich (32). Recent investigations, many of which have been sponsored in whole or in part by CRC, have been reviewed especially for the Council by J. W. Clark and H. N. Hill (33).

Interesting and important as these studies of lateral-torsional buckling may be, the reader should keep in mind that in the great majority of practical design situations, adequate lateral support is provided and the whole problem of lateral buckling is thus usually avoided. The open rolled I, WF, and channel sections are very efficient and strong when loaded through the shear center in a plane parallel with the web. These same sections are relatively weak in either torsion or lateral bending, and these weaknesses accentuate the design problem that arises when they are not supported laterally. The overall design problem may include that of combined bending and torsion (if the sections are not loaded through the shear center) and biaxial bending (if the load is not parallel with the web). Steps taken by the designer to improve resistance to lateral-torsional buckling will also improve the efficiency of the beam with respect to the other two aforementioned problems.

The solution of the lateral buckling problem for the rectangular beam deserves first attention because it is relatively simple and it may be used directly in the design of box girders. As mentioned in the introduction, box girders should be used in important applications where long and/or heavily loaded spans must be designed without lateral bracing.

For a rectangular section under uniform bending moment and with simple end supports the critical maximum stress at buckling is given by the formula:

$$\sigma_c = \frac{\pi}{S L} \sqrt{J G E I_y} \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad (4)$$

Eq. 4 for the rectangular section may be used for box girders of rectangular cross section, with continuous closed cross-section achieved by welded or riveted connections along all longitudinal seams (Fig. 6). For substitution in Eq. 4, the following is a close approximation of the section constant "J," based on Bredt's (35) for box sections:

$$J = \frac{2\,b^2\,d^2}{\frac{b}{t_f} + \frac{d}{t_w}} \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad (5)$$

The critical "elastic" lateral buckling stress of box girders of usual proportions is far above the yield point and failure will usually be by local buckling or by inelastic lateral buckling.

In the design of box girders, there is little need to tabulate various effective lengths as determined by end restraint and loading conditions, since there is little or no stress reduction needed below values permitted for full restraint in any case.

6. Attention is given to simple procedures for approximating the lateral buckling strength of metal beams in the inelastic range.

As in the case of the column, the strength of a laterally unsupported beam of relatively small length will be determined by inelastic rather than elastic behavior. Like the column, too, the very short beam may be expected to develop full yield strength of the material.

A method of estimating the effect of inelasticity on the strength of beams and girders is to assume that the relationship between the elastic and inelastic buckling strength is the same for beams as it is for columns. Inelastic buckling strength of beams may then be estimated from a column strength curve. This procedure is applicable to aluminum alloy members since the tangent modulus buckling curve agrees well with tests on both columns and beams (36). Likewise, in the case of steel, the tests by Hechtman, Hattrup, Styer, and Tiedemann (60) are in good agreement (generally conservatively so) with the predictions of the basic column strength curve proposed herein in Fig. 1. This approach was validated in the case of I-beam tests as shown in Fig. 7 and was suggested by Bleich (32, p. 130), for the more general beam-column problem. It is also used by the Aluminum Company of America (37, p. 110).

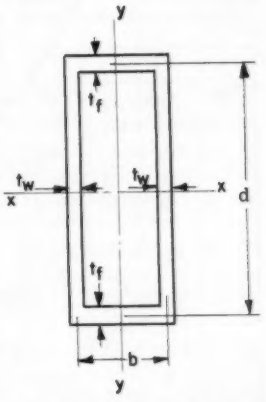

FIG. 6.—BOX GIRDER CROSS SECTION

The basic column strengths in Fig. 1 are determined by Eq. 2 which may be written in either of two ways:

$$\sigma_{c\,Col} = \sigma_y \left[1 - \frac{\sigma_y}{4\,\pi^2\,E} \left|\frac{K\,L}{r}\right|^2\right]$$

or

$$\sigma_{c\,Col} = \sigma_y \left[1 - \frac{\sigma_y}{4\sigma_e}\right]$$

where σ_e is the Euler stress. In the second form, it may be assumed to be applicable to a beam by the substitution of σ_{cb} for the beam in place of the Euler buckling stress, in which case,

$$\left|\frac{K\,L}{r}\right|_{equiv} = \pi\sqrt{\frac{E}{\sigma_{cb}}} \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad (6)$$

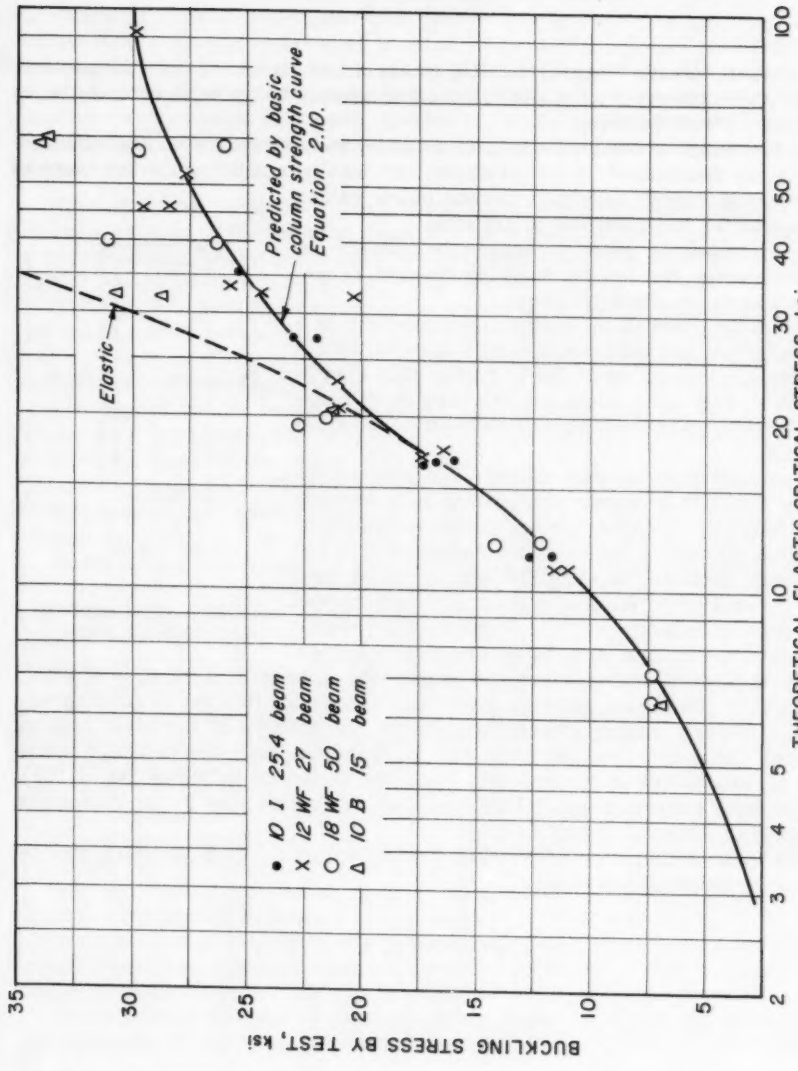
BUCKLING STRESS BY TEST, ksi
THEORETICAL ELASTIC CRITICAL STRESS, ksi
Elastic
Predicted by basic column strength curve Equation 2.10.
• 10 I 25.4 beam
X 12 WF 27 beam
○ 18 WF 50 beam
△ 10 B 15 beam

FIG. 7.—BUCKLING STRESS BY TEST VS. CRITICAL STRESS, BASED ON MEASURED PROPERTIES.

For the rectangular or box section, σ_{cb} may be determined from Eq. 4 and substituted in Eq. 6, or a more direct approach may be made. In Eq. 4 replace G by its equivalent

$$G = \frac{E}{2(1+\nu)}$$

If ν (Poisson's ratio) is taken as 0.318 (a compromise between 0.30 for steel and 0.33 for aluminum alloy) the introduction of Eq. 4 into Eq. 6 yields:

$$\left|\frac{K\,L}{r}\right|_{equiv} = \sqrt{\frac{5.1\,L\,S_x}{\sqrt{J\,I_y}}} \quad \ldots\ldots\ldots\ldots\ldots \quad (7)$$

Eq. 7 is applicable to either aluminum alloy or steel and the approximation introduced for Poisson's Ratio has much less effect than the normal variation in mechanical properties of the materials. Eq. 7 is limited in application to box or rectangular sections. Space does not permit the inclusion of illustrative examples of the application of these procedures, such as are included in the complete Guide.

7. Recognition of the fact that L/b as a parameter, long familiar in design formulas for laterally unsupported beams, still has a specific place in certain areas of design in spite of the advantageous introduction of the more recent (L d)/(b t) formula.

The truth of this statement was brought forcibly to the attention of CRC by Neil van Eenam, Bridge Engineer, Bureau of Public Roads, when he showed several years ago, in an unpublished study, that the application of the (L d)/(b t) formula to short unsupported segments of deep bridge girders or floor beams would reduce permissible stresses considerably below levels that had proved satisfactory over many years of design application. In recognizing this same fact, George Winter has pointed out, as a result of his studies (23), that L/b (or L/r_Y) is the controlling parameter "in the case of deep and/or thin-walled beams" whereas (L d)/(b t) is applicable "in the case of shallow and/or thick-walled beams." The Guide clearly differentiates areas of design applicability of the two parameters, and, in addition, provides somewhat more complex formulas that incorporate both effects in a single equation.

The essential reason for the foregoing demarcation is the fact that lateral buckling failure of a beam always involves lateral bending and twisting in combination. The term L/b is a measure of lateral bending flexibility, which is paramount in the lateral buckling of deep and/or thin-walled members. The term (L d)/(b t) is primarily a measure of torsional flexibility, which is a primary factor in lateral buckling of stocky and/or shallow members of which WF and I beams commonly used in building construction are typical.

The (L d)/(b t) design formula of AISC Specifications is practically synonymous with the name of Karl de Vries (38) who first demonstrated its simplicity and applicability. It is strictly applicable only to the double symmetric WF or I beam and depends on certain simplifying assumptions which are explained in the Guide as follows.

Three alternate forms of Eq. 8 closely approximate the critical elastic buckling stress due to bending for I, WF, roller sections, or double symmetric plate girders, when loaded by end couples in the plane of the web, or by transverse loads applied at the centroidal axis in the plane of the web:

$$\sigma_c = \frac{C_1 \pi \sqrt{E I_y G J}}{S (K L)} \sqrt{1 + \frac{\pi^2 a^2}{(K L)2}} \quad \text{(8a)}$$

$$\sigma_c = \frac{C_1 \pi \sqrt{E I_y G J}}{S (K L)} \sqrt{1 + \frac{\pi^2 C_w E}{J G (K L)2}} \quad \text{(8b)}$$

$$\sigma_c = \frac{C_1 \pi^2 E I_y h}{2 S (K L)^2} \sqrt{1 + \frac{(K L)^2 J G}{\pi^2 C_w E}} \quad \text{(8c)}$$

Eqs. 8a and 8b simply involve an interchange of usage of the parameters "a" and C_w which are interrelated for the I and WF shape as follows:

$$a = \sqrt{\frac{C_w E}{J G}} = \frac{h}{2} \sqrt{\frac{E I_y}{J G}} \quad \text{(9)}$$

The term C_w is the torsion warping constant, not limited to any particular shape (See Ref. 32, p. 120, where C_w appears as Γ). The term a is referred to as the torsion bending constant and is listed numerically for rolled steel WF and Γ shapes in Bethlehem Steel Company Booklet S-57. The term C_w is listed for rolled aluminum shapes in the "Alcoa Structural Handbook" where it is noted as C_S.

The torsion constant J may be approximated by $\Sigma\ (1/3)\ b\ t^3$ for all shapes made up of component rectangular parts. Tabulated values of J for rolled steel WF and I beam shapes are listed in Bethlehem Steel Company Booklet S-57 as "K" and are listed for aluminum alloy shapes as "J" in the "Alcoa Structural Handbook."

In Eq. 8a and 8b, the radical approaches unity as a/L decreases for shallow and/or stocky beams and girders, whereas, for deep and/or thin walled girders, the radical in Eq. 8c approaches unity as L/a decreases.

The term C_1 depends primarily on the distribution of bending moment and the end support and restraint conditions. Graphs and tables are provided in the Guide for its accurate determination—however, for the typical cases of uniform load applied to a shallow beam or uniform moment, it may be taken as unity.

Rolled wide-flange and I beam sections that are used as laterally unsupported beams usually have a depth of at least twice the flange breadth and a web thickness at least 2/3 the flange thickness. Using these as typical proportions, the following approximate formulas for section properties result:

S_x	$= 1.1\ b\ t\ d$	L = laterally unsupported span
I_y	$= \frac{b^3 t}{6}$	b = flange width
J	$= 0.9\ b\ t^3$	d = depth of beam
h	$= 0.95\ d$	t = flange thickness
a	$= \frac{h}{2} \sqrt{\frac{E I_y}{J G}} \approx 0.33 \frac{d\ b}{t}$	h = distance between flange centroids

If, for $C_1 = K = 1$, the foregoing substitutions are made in Eq. 8, the following simplified formulas are obtained:

$$\sigma_c = \frac{0.69\,E}{\frac{L\,d}{b\,t}} \sqrt{1 + 1.07\,\frac{(d/t)^2}{(L/b)^2}} \quad \ldots\ldots\ldots (10a)$$

$$\sigma_c = \frac{0.71\,E}{\left|\frac{L}{b}\right|^2} \sqrt{1 + 0.93\,\frac{(L/b)^2}{(d/t)^2}} \quad \ldots\ldots\ldots (10b)$$

In each of the foregoing, it is noted that two parameters, L/b and d/t, govern the result. In Eq. 10a, the part outside of the radical is similar in form to the current AISC design formula.

Referring back to Eq. 10a of the preceding section, it may be seen that in the case of shallow and/or stocky beams the radical may be taken as equal to unity to provide a conservative estimate by the approximate formula:

$$\sigma_c = \frac{0.69\,E}{\frac{L\,d}{b\,t}} \quad \ldots\ldots\ldots\ldots\ldots\ldots (11)$$

Similarly, from Eq. 10b, in the case of deep and/or thin walled beams, a satisfactory approximation is:

$$\sigma_c = \frac{0.71\,E}{\left|\frac{L}{b}\right|^2} \quad \ldots\ldots\ldots\ldots\ldots\ldots (12)$$

These two equations give identical results when

$$\frac{L}{b} = 1.035\,\frac{d}{t}$$

Since either equation gives a conservative estimate (for the special case of uniform bending moment and simple end supports) the equation that gives the larger value of σ_c should be chosen, and the following approximate rule may serve as a guide:

Use Eq. 11 when L/b is greater than d/t.

In the case of application to thin-walled construction, as covered by the AISI Specifications (21), only Eq. 12 is applicable, and the foregoing substitutions for S_x and I_y lead to the following modification of Eq. 12:

$$\sigma_c = \left[\frac{0.71\,E}{(L/r_y)^2}\right] \left[3.3\left(\frac{d}{r_x}\right)^2\right] \quad \ldots\ldots\ldots\ldots (13)$$

For most structural I shapes, $r_x \approx 0.38d$, in which case

$$\sigma_c \approx \frac{16\,E}{\left|\frac{L}{r_y}\right|^2} \quad \ldots\ldots\ldots\ldots\ldots\ldots (14)$$

This modification permits handling of doubly-symmetric sections with non-rectangular flanges. For example, the edges of the flanges may be lipped, as is common in thin-walled construction.

In the case of rolled steel I and WF shapes of reasonably long span, L/b will usually be greater than d/t, thus justifying in this particular range of application the current AISC Specification design formula of

$$F_b = \frac{12{,}000}{\frac{L\,d}{b\,t}} \quad \text{(ksi)} \quad \ldots\ldots\ldots\ldots\ldots \quad (15)$$

8. Recognition and delineation of areas wherein the Ld/bt formula should not be applied, that is, to unsymmetrical sections, channel sections, beams with variable moment of inertia, etc.

It is with no intent to discredit the important areas of application of the de Vries (L d)/(b t) formula that the Guide defines other, less important areas of design, for which the (L d)/(b t) formula is not well suited.

The beam section unsymmetrical about y-y has been reviewed in detail in another symposium paper by Clark and Hill (33). For the special case of the rectangular shaped flange, with the area of the compression flange greater than that of the tension flange, as is usual in this type of design problem, the Guide presents an equation of the following type that has been found to give good agreement with the exact solution:

$$\sigma_c = \frac{0.48\,E}{\left(\frac{L}{b}\right)^2}\left[0.6 + \sqrt{0.75 + \frac{\left(\frac{L}{b}\right)^2}{\left(\frac{d}{t}\right)^2}}\right] \quad \ldots\ldots\ldots \quad (16)$$

$$(A_c > A_t)$$

In Eq. 16, b and t refer to the compression flange only.

Alternatively in terms of r_y for the compression flange only

$$\sigma_c = \frac{5.75\,E}{\left(\frac{L}{r_y}\right)^2}\left[0.6 + \sqrt{0.75 + \frac{\left(\frac{L}{r_y}\right)^2}{12\left(\frac{d}{t}\right)^2}}\right] \quad \ldots\ldots\ldots \quad (17)$$

$$(A_c > A_t)$$

In this equation σ_c is independent of the amount of tension flange area. Nevertheless, over the extreme range of sections that has been used as a basis for the comparison of various formulas by Clark and Hill (33), Eq. 16 or 17 give results generally as good or better than the more complex approximate formulas, especially for ratios of A_c/A_t of about 2. The results for σ_c are generally conservative and reach as much as 35% on the low side as the tension flange diminishes in the limit to zero area.

The parameters in Eq. 16 or 17 are similar to those of Eq. 10, for doubly symmetric sections. Approximately the same values are obtained when the second term under the radical in each of these equations is put equal to zero, thus neglecting the torsional rigidity.

The exceptional case, when the tension flange has more area than the compression flange, is covered in the Guide by a safe and accurate approximate formula developed by George Winter (34).

In the case of doubly symmetric girders with variable flange area, there will usually be intermittent lateral supports and the lateral buckling design check will refer to a segment in which constant cross-section may be assumed. However, during construction and prior to introduction of lateral bracing, or in some other situation wherein no lateral support is provided, a reasonable estimate of girder strength may be made by means of the following:

$$\sigma_c = \sigma_{c(u)} \sqrt{\frac{I_{min}}{I_{max}}} \quad \ldots\ldots\ldots\ldots\ldots \quad (18)$$

where $\sigma_{c(u)}$ is estimated for a section of constant $I = I_{max.}$ by Eq. 10. Eq. 18 is an empirical simplification of the results of preliminary studies by W. J. Austin in which an idealized loading was assumed of a type that would produce at every cross-section the same maximum fiber stresses in the flanges. Thus, the state of stress is comparable to that produced by constant moment in a beam of constant cross-section. More information is needed on this particular subject. However, the proposed design rule is simpler and more conservative than similar proposals made in connection with BS 153 in Great Britain.

If laterally unsupported channel beam sections are loaded and supported vertically by end reactions that pass through the centroidal axis, they will twist as well as bend, except for the special case wherein the loads are directed normal to the plane of the web so as to cause bending in the weakest plane. For such loadings, other than the special case mentioned above, the design problem is one of stress due to combined bending and torsion. If, however, a channel is loaded and supported by resultant forces that pass through the shear center, it will bend without twisting. If, further, the loads are parallel with the plane of the web, bending will be in the same plane and lateral-torsional buckling may pose a design limitation if continuous lateral support is absent.

Hill (39) has shown that the use of formulas obtained for the I shaped section will provide an approximation for the channel that in no case errs by more than 6%. However, the error is on the unsafe side, and it is recommended that for rolled channel sections, loaded at points by relatively rigid and non-rotating connections, as described above, Eq. 4 be used as a simple procedure in determining the allowable stress. Thus, no advantage would be taken of warping rigidity and the unsupported lengths of channels between load points of the type indicated may be safely designed by Eq. 4. The use of Eq. 4 requires the use of the torsion constant J, listed for rolled aluminum shapes in the Alcoa Structural Handbook or easily approximated for rolled steel channels by $\Sigma (1/3)\ b\ t^3$. Alternatively, if a more precise determination is desired, and the warping constant C_w known, the effects of warping constraint may be included by use of Eq. 8b.

Chapter 4 of the Guide concludes with a discussion of proper lateral support for beams. In the great majority of practical design situations, adequate lateral support is provided and the whole problem of lateral buckling is thus usually avoided.

9. The Guide assesses the usefulness of the initial yield criteria as a basis for the design of beam-columns.

The term "beam-column" denotes a member subjected not only to axial force, but also to moments resulting from transverse forces and/or from a known eccentricity of the axial force at one or both ends. As the bending moments approach zero, the member tends toward a centrally loaded column, as discussed under Sections 1 and 2. As the axial force approaches zero, the

problem becomes that of a beam, which, for the laterally unsupported case, involves the buckling and design problems that have been treated in Sections 5, 6, 7, and 8.

A number of studies and reviews of the beam-column problem have been sponsored by CRC (40,41,42). Specifically for the present symposium as well as for the Guide, summary reports on this subject (43,44) have been prepared by Walter J. Austin.

When torsion is not a problem, two simple approximate methods have been developed for estimating the strength of beam-columns. The first method is based upon the concept that a lower bound to the collapse load is provided by the load which produces a hypothetical initiation of yielding in the fibers subjected to maximum stress. This method is limited to members that fail by bending in the plane of the applied loads. The second approach is by use of the interaction formulas to be discussed in Section 10.

It should be noted that if the initial yield theory is to be used to estimate the strength of columns with small lateral loads or small intentional eccentricities, it is necessary to assume that some unintentional "equivalent" eccentricity exists, in order that the "initial yield" strength curve approach a proper limit for concentrically loaded columns. The effects of all imperfections such as initial crookedness, non-homogeneity, residual stresses, etc., are lumped into the equivalent initial eccentricity. Thus, prior to the recognition of the importance of residual stress, the use of the "secant formula" for the eccentrically loaded column was a plausible and valuable device in the explanation of centrally loaded column behavior, and became the basis for the current AREA and AASHO formulas for both central (concentric) and eccentric loads. In the former case, a simple parabolic formula (within the range of L/r less than 120) provides a good approximation of the secant formula with an implied allowance for initial eccentricity.

The first step in developing the initial yield criterion is the analysis of maximum stress due both to the applied bending moment and the deflection. In the stress analysis procedure, applied to design, the combination of loads that will cause the maximum combined stress to reach the yield point is divided by the desired factor of safety to give the working combination of loads. The special case of this procedure involving eccentric end loads, with equal end eccentricities, gives what has been called the secant formula. Thus,

$$f_m = \frac{P}{A}\left[1 + \left(\frac{e\,c}{r^2} + \frac{e_o\,c}{r^2}\right) \sec \frac{L}{2\,r}\sqrt{\frac{P}{A\,E}}\right] \quad \ldots\ldots (19)$$

where f_m denotes the maximum stress (at midspan), e is the applied eccentricity of the load resultant, and e_o is the equivalent eccentricity due to defects, etc.

The limiting average stress for design may be obtained by replacing P by n P, where n is the factor of safety, by setting $f_m = f_y$, and by solving for P/A. Thus, the permissible average stress P/A is given by

$$\frac{P}{A} \leq \frac{f_y/n}{\left[1 + \left(\frac{e\,c}{r^2} + \frac{e_o\,c}{r^2}\right) \sec \frac{L}{2\,r}\sqrt{\frac{n\,p}{A\,E}}\right]} \quad \ldots\ldots\ldots (20)$$

The secant formula (Eq. 20) applies to only a particular loading condition and cannot, by itself, furnish the basis for a comprehensive design procedure.

A more comprehensive incipient-yield-type procedure for the design of eccentrically loaded columns may be based upon the solution for unequal eccentricities. This solution was given by D. H. Young (45) and is discussed fully by Timoshenko (46). An elaborate extension of this approach, including consideration of both eccentricity and curvature, with a variety of end loading and restraint conditions, has been developed by H. K. Stephenson (47,48). Tables and charts are included in great detail to minimize the inherent difficulties in direct application of the complex formulas that are involved.

O. G. Julian (49) in the Proceedings of the Fourth Technical Session, CRC, outlines a procedure, using a convenient nomogram, by which the maximum fiber stresses at any section in a compressed member with end moments can be found.

Proponents of the initial yield criterion point to the long record of successful application of the secant formula in specifications for the design of steel railway and highway bridges: for the built-up box type compression chord, not susceptible to torsional buckling, with unavoidable end eccentricities entering due to lack of axiality of load or because of secondary stresses; and for reasonably thin material that is likely to buckle at or near the yield point. Under these conditions, the secant formula, made usable by means of charts or tables, is a proved and adequate approach to design.

Nevertheless, in summary, four objections to the initial yield criterion approach to beam-column design may be restated:

(1) Its application lacks generality as to material, since its use should be limited to materials exhibiting linear elasticity up to a sudden "yield point." The effect of residual stress cannot rationally be taken into account.

(2) Even for annealed steel, in the case of certain ranges of the pertinent parameters, such as bending about the minor y-y axis with large eccentricities, the design on the basis of initial yield may be grossly over-conservative.

(3) For the I-shaped column that is bent about the major (x-x) axis and laterally unsupported in the weak direction, the neglect of the lateral buckling tendency may render the initial yield criterion significantly unconservative.

(4) The effect of end restraint cannot be rationally introduced into the secant formula analysis since the procedure is based on the analysis of an unrestrained member.

The interaction formulas to be discussed in Section 10 provide alternative approaches to beam-column design that have greater generality and that overcome the first three of the foregoing objections to the initial yield procedure. In the range of application where the initial yield approach is applicable, the interaction formula will yield similar design results.

10. The Guide evaluates the adequacy of the interaction formulas as applied to beam-column design.

The strength of columns subjected to combined flexure and compressive axial load and constrained to bend without twist in the plane of the loads and centroidal axis can be expressed very conveniently in terms of the ratios P/P_u and M/M_u where P is the thrust at failure, and P_u is the collapse load for the column centrally loaded for buckling in the unconstrained plane. The term M is the actual maximum bending moment for no axial load. It is possible to write simple empirical interaction formulas in terms of force or equivalent stress ratios to express the strength of all practical types of beam-columns under all common loadings.

For beam-columns the following simple straight line equation is the basis for several interaction equations:

$$\frac{P}{P_u} + \frac{M}{M_u} = 1 \quad \ldots\ldots\ldots\ldots\ldots\ldots (21)$$

In the elastic range a rough approximation of the maximum bending moment for simply supported columns of uniform section subjected to any variation of bending forces, where the maximum moment occurs at or near the center of the member, is given by

$$M = \frac{M_o}{\left(1 - \frac{P}{P_e}\right)} \quad \ldots\ldots\ldots\ldots\ldots\ldots (22)$$

where P_e is the Euler elastic critical load in the plane of applied moment and M_o is the initial maximum moment, not considering the moment due to the axial load interacting with the deflections. For this case, Eq. 21 becomes

$$\frac{P}{P_u} + \frac{M_o}{M_u\left(1 - \frac{P}{P_e}\right)} = 1 \quad \ldots\ldots\ldots\ldots\ldots\ldots (23)$$

For eccentrically loaded columns, where e is the eccentricity, equal at both ends, Eq. 23 takes the following form:

$$\frac{P}{P_u} + \frac{P_e}{M_u\left(1 - \frac{P}{P_e}\right)} = 1 \quad \ldots\ldots\ldots\ldots\ldots\ldots (24)$$

Shanley (50) has discussed interaction formulas of this type in considerable detail, and was an early proponent (51) of this approach to design. In Reference (51) Shanley reviews early developments in the aircraft design field.

The following simpler form of the interaction equation,

$$\frac{P}{P_u} + \frac{M_o}{M_u} = 1 \quad \ldots\ldots\ldots\ldots\ldots\ldots (25)$$

has been used as the basis for some design specifications (22). When L/r is large and the end eccentricity or applied moment small but maximum near mid-height, Eq. 25 may overestimate the carrying capacity of the column. This fact has been demonstrated by tests reported by Hill, Hartmann and Clark (52,53), by Mason, Fisher and Winter (55), and is further discussed herein in connection with Table 1 (to be presented subsequently).

In the case of the laterally unsupported beam-column of I-shaped sections bent in the principal plane of maximum strength, Hill and Clark (56) have found that the modified interaction formula, Eq. 26, is quite satisfactory for both the elastic and inelastic range.

$$\frac{P}{P_u} + \frac{M_o}{M_{ul}\left(1 - \frac{P}{P_{el}}\right)} = 1 \quad \ldots\ldots\ldots\ldots\ldots\ldots (26)$$

The terms (of Eq. 26) have essentially the same meaning as in Eq. 23, except that P_u and M_{ul} now reflect the possibility of weak plane buckling or lateral-torsional buckling, respectively. It should be noted that P_{el} is the elastic buckling load in the strong plane.

Eq. 26 is conservative and shows good agreement with the test results for eccentrically loaded aluminum columns that fail by lateral-torsional buckling. The simpler straight-line interaction formula

$$\frac{P}{P_u} + \frac{M_o}{M_{ul}} = 1 \quad \ldots\ldots\ldots\ldots\ldots\ldots (27)$$

gives results with greater scatter and overestimates the strength of slender members for which the first term P/P_u is significantly large in comparison with M_o/M_{ul}. Campus and Massonnet (56) also found from their tests of long steel columns that Eq. 26 agrees with test data better than Eq. 27 and that Eq. 27 gives a high estimate of strength in some cases. Consider now design rather than strength.

Interaction formulas are well suited for design use. Such formulas have a simple form, are convenient to use, and have a wide scope of application. By their very nature interaction formulas lead to allowable stresses which vary gradually and continuously in a smooth transition from concentrically loaded columns to beam-columns.

In this report are considered the straight line interaction design formula, Eq. 28, used in the AISC Specification:

$$\frac{f_a}{F_a} + \frac{f_b}{F_b} \leq 1 \quad \ldots\ldots\ldots\ldots\ldots\ldots (28)$$

and the modified interaction formula proposed by Clark (53):

$$\frac{f_a}{F_a} + \frac{f_b}{F_b\left(1 - \frac{n f_a}{f_e}\right)} \leq 1 \quad \ldots\ldots\ldots\ldots\ldots (29)$$

recommended previously in a report by Committee E, CRC, (40). In these formulas (Eq. 28 and 29),

f_a = P/A = average compressive stress;

F_a = allowable average compressive stress for concentrically loaded member, as determined by maximum slenderness ratio (regardless of plane of applied moment);

f_b = M/S = maximum compressive stress caused by flexure, neglecting the additional moment due to the axial load interacting with the deflections;

F_b = allowable compressive bending stress for member with no thrust;

$f_e = \frac{P_e}{A}$ = average compressive stress at the Euler load;

n = factor of safety.

Interaction formulas for design are devised to give a desired factor of safety against some limiting condition which may be either the initiation of yielding or collapse. The nature of an interaction formula obviously depends upon the expressions chosen for the allowable compressive and bending stress, F_a and F_b. The bending stress F_b is usually taken as the yield stress divided by the factor of safety, reduced, if necessary, if lateral torsional buckling is a consideration. The allowable axial compressive stress F_a may be considered to be chosen to give either a proper factor of safety against collapse or against

initial yielding, taking into account possible residual stress, end eccentricities, etc., which are appropriate for the intended use of the column which is not designed for specific bending moments.

In discussing formulas that are essentially empirical, as is the case for either Eq. 28 or 29, it is understandable that some difference of opinion may arise. Both of these equations have found application in current design specifications. Unquestionably, Eq. 29 is the more conservative, and it does take rough account of additional moments caused by deflection, whereas they are ignored in Eq. 28. Since, at any given level of stress, the deflection produced by bending is directly proportional to E, Eq. 28 is better adapted to the design of steel columns than it would be for aluminum alloy columns.

Eq. 28 is much simpler to use in actual design application than Eq. 29, much simpler, in fact, than a superficial evaluation of the arithmetic processes would indicate. Each interaction formula represents an interpolation between identical extreme limits of behavior for the limiting case of beam or column and in the limit each yields identical design criteria that are as valid as they can be made to be for the intended use.

In order to provide further perspective, the Galambos-Ketter (57) interaction curves for ultimate strength were used as a basis for the following tabulation of factors of safety resulting from use of the simpler formula, Eq. 28.

TABLE 1.—FACTOR OF SAFETY WITH RESPECT TO BEAM COLUMN STRENGTH, USING EQ. 28, WITH F_a FROM FIG. 1 (STRUCTURAL STEEL) WITH A FACTOR OF SAFETY OF 1.95 FOR CENTRAL LOADING.

ec/r^2	Slenderness Ratio L/r				
	20	50	80	120	160
0.1	1.84	1.87	1.87	1.81	1.87
1.5	1.98	1.82	1.69	1.55	1.73
1.0	1.89	1.81	1.66	1.52	1.66
3.0	1.96	1.85	1.72	1.57	1.63
5.0	1.97	1.86	1.78	1.65	1.64

Existing practice in steel building design calls for factors of safety ranging from 1.65 to 1.90. The deviations from an assumed average of 1.80 in Table 1 are not more than 8% for L/r up to and including 80. For L/r of 120 the maximum deviations of 15% on the unconservative side are reached. Thus, there is a range for which the simpler formula is adequate, but there is also a range where it is deficient. If the simpler formula is used in all cases, it must be on the basis of a conservatism in other design assumptions, such as, for example, conditions of end restraint, making the "real" factor of safety greater than the "design" factor of safety.

It should be emphasized that when the maximum moment is not at or near the center of the span, both Eqs. 28 and 29 are excessively conservative. The Guide presents existing recommendations of Horne (58) and Massonnet (56) to take care of such situations.

Since the maximum stress may occur at the ends, the following condition must be satisfied in addition to the requirements of the appropriate interaction formula:

$$\frac{f_a}{F_{ao}} + \frac{M_m}{S F_b} \leq 1 \quad \ldots\ldots\ldots\ldots\ldots\ldots (30)$$

where M_m is the maximum moment (at the end) and F_{ao} is the allowable stress for concentrically loaded columns with $L/r = 0$.

Interaction formulas are especially convenient for cases of bending about both axes. For such application the modified interaction formula becomes,

$$\frac{f_a}{F_a} + \frac{f_{b1}}{F_{b1}\,(1 - n\, f_a/f_{e1})} + \frac{f_{b2}}{F_{b2}\,(1 - n\, f_a/f_{e2})} \leq 1 \ldots . (31)$$

The straight line interaction formula for bending about both principal axes is,

$$\frac{f_a}{F_a} + \frac{f_{b1}}{F_{b1}} + \frac{f_{b2}}{F_{b2}} \leq 1 \quad \ldots\ldots\ldots\ldots\ldots\ldots (32)$$

In Eqs. 31 and 32 it should be noted that F_a, which is the allowable axial stress for the concentrically loaded column, is determined from the maximum slenderness ratio, L/r_2. Also, F_{b1}, which is the allowable compressive stress for bending, may be determined from the lateral buckling strength of the member as a beam. Therefore, the value of F_{b1} is a function of the length and cross section dimensions (such as $(L\,d)/(b\,t)$) and may be much smaller than the corresponding allowable stress due to bending if the beam is restrained against lateral and twisting failure. Hence, F_{b2} is a constant usually equal to the yield point divided by the factor of safety. The parameters f_{e1} and f_{e2} in Eq. 31 are the critical average stresses for elastic buckling about the major and minor principal axes, respectively.

The above interaction type formulas are suitable for the most severe loading, namely equal eccentricities at both ends in one or both principal directions. These formulas are also very good for transverse loading which produces a symmetrically shaped moment diagram with the maximum near mid-height. However, for less severe cases, such as oblique eccentric loadings, the factor of safety increases considerably. For greater economy and to maintain a more uniform factor of safety, the modifications suggested by Massonnet or Horne may be used. That is, the bending stresses in the two directions, f_{b1} and f_{b2} may each be calculated from equivalent uniform moments. For oblique loading it is also necessary to check the stresses at the ends, especially if the eccentricities are equal and opposite and the column is stocky. The following condition must be satisfied at each end:

$$\frac{f_a}{F_{ao}} + \frac{f_{b1}}{F_{b1}} + \frac{f_{b2}}{F_{b2}} \leq 1 \quad \ldots\ldots\ldots\ldots\ldots (33)$$

where F_{ao} is the allowable stress for a concentrically loaded column with $L/r = 0$ and where f_{b1} and f_{b2} are the maximum bending stresses in the two principal directions.

11. The Guide recommends the application of the effective length concept in the design of the framed beam column.

Most beam-columns do not occur as isolated, simply-supported members but as parts of frameworks, often with rigid connections. In this case a really rational design procedure should take account of the actual end restraints afforded by the contiguous members. One way to do this would be to increase the actual loads by appropriate factors of safety and then to make an ultimate strength analysis to find the loads that produce the chosen limiting stress. In such an analysis it would be necessary to consider the effect of the axial load on the stiffness constants in any frame analysis and the effect of the axial load on the deflections and moments in individual members. Such a procedure would generally be quite laborious. For practical design the work of George Winter and associates at Cornell, done under joint CRC and Bureau of Public Road sponsorship, has indicated a simple approach which is conservative, but less conservative than neglecting altogether the strength afforded by the end restraints of adjacent members.

George Winter discussed the means to do this at the Fourth Technical Session of the Column Research Council, and the following remarks are quoted from his paper (59):

> "For eccentrically loaded, end-restrained columns Dr. Annabel Lee and Professor P. P. Bijlaard have shown, respectively for initial yield and for ultimate strength, that a safe design results from the following method: The restrained column is replaced by an equivalent, hinged column whose length is equal to the effective length for concentric compression of the real, restrained column, and this equivalent column is analyzed for compression plus that part of the total end moments which is resisted by the column alone.
>
> "They have also shown that in order to determine that part of the total joint moment which is resisted by the column it is sufficiently accurate to use elementary moment distribution, discounting the effect of axial force on stiffness.*
>
> "This simplified procedure could result in the following:
>
> "(i) For rigid frames, determine the end moments of the column by conventional moment analysis, without regard to change of effective stiffness caused by axial load. Then dimension the column for the normal force and the moments so obtained, but use kL instead of the real length L in determining the slenderness ratio. This method is applicable regardless of the provision specified for compression plus bending in the particular code (secant formula, interaction formula, or any other rational device acceptable in the design specification).
>
> "(ii) For real eccentricities, such as in truss members with joint eccentricities, multi-story columns with offset axes, etc., which result in end moments M_e, obtain the effective eccentricities of all compression members abutting to a given joint from elementary moment distribution, i.e., from,

$$e_n = \frac{M_e}{P_n} \quad \frac{K_n}{\Sigma K} \quad ** \qquad \ldots\ldots\ldots\ldots\ldots\ldots (34)$$

---

* A conservative approximation, since it will result in an over estimate of the bending moment.

** The notation of Ref. 59 is used.

where e_n is the effective eccentricity, P_n the axial compression force, and K_n the rigidity I/L of compression member "n". The sum ΣK includes the rigidities of all members connected to the given joint. Then dimension the member for simultaneous action $N = P_n$ and $M = P_n e_n$, using kL for the length of the member in determining its slenderness ratio."

CONCLUSION

Thus concludes a digest of eleven "highlights" of the forthcoming edition of the Column Research Council "Guide." It is anticipated that the subjects of plate girder webs and pony truss bridges will be added in a future second edition of the Guide. Research in these areas has been reported upon at this symposium. Undoubtedly, improvements also will be made in the material that is now about to be published. There are still areas to be clarified and more research is needed as the engineer proceeds further toward the paradoxically difficult goal of achieving design criteria that combine rationality with simplicity.

APPENDIX I.—NOTATION

A = area of cross section;

a = distance;

a = torsion-bending constant for I-section;

b = flange width;

b_e = effective width of plate;

C_w = torsion warping constant;

c = distance;

d = depth of section;

E = elastic stress-strain modulus;

E_t = tangent modulus;

e_o = equivalent eccentricity due to imperfections;

F_a = permissible column stress under central load;

F_b = permissible normal stress due to bending;

f_a = P/A;

f_b = M/S;

f_e = equivalent design stress on effective width;

f_m = maximum normal stress;

G = elastic shear modulus;

I_x, I_y = moment of inertia about xx, yy, axes, respectively;

J = torsion constant;

k = coefficient in plate buckling;

K = effective length factor for column;

ksi = kips per square inch;

L = length of member;

M_c = critical bending moment;

M_o = bending moment without regard to added moment caused by deflection;

M_u = ultimate bending moment (no axial load);

M_x, M_y, M_z = bending moment about x, y, or z axes, respectively;

n = factor of safety;

P = resultant force, positive if compression;

P_e = Euler (elastic) column load;

P_u = column collapse load, centrally loaded;

psi = pounds per square inch;

r = radius of gyration of cross section;

r_o = radius of gyration of component part of cross section;

S = section modulus;

t = flange thickness; or thickness of a single plate;

x, y, z = coordinate axes or distances;

xx, yy = centroidal axes;

δ_o = deflection without regard to added moment caused by deflection;

ν = Poisson's ratio;

σ = normal stress;

σ_c = critical stress;

$\sigma_e = \frac{P_e}{A}$;

σ_{cb} = elastic buckling stress for a beam;

σ_{RC} = maximum residual stress in compression;

σ_y = yield strength (aluminum) or (yield level) steel, as determined by elastic range offset;

σ_y = ASTM specification yield point in tension;

τ = shear stress;

$\tau = \frac{E_t}{E}$; and

τ_e = critical shear stress.

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PRELIMINARY ANALYSIS OF CONTINUOUS GABLE FRAMES

By James W. Gillespie,[1] A.M. ASCE and Jan J. Tuma,[2] F. ASCE

SYNOPSIS

Charts for the preliminary analysis of continuous gable frames are presented. The study is restricted to two, three, and four equal-span frames of constant cross section subjected to uniformly distributed loads. All relationships are based on the assumption of elastic deformation. Numerical examples are included.

INTRODUCTION

The design of continuous gable frames is a cut and trial procedure. The sizes of members, which cannot be found without knowing them first, must be assumed. Then the frame can be analyzed and redesigned in several trials, until a satisfactory solution is found.

To facilitate the selection of member sizes and to accelerate this process of successive approximations, algebraic formulas, based on the principle of elastic deformation, have been developed for single-span gable frames by A.

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[1] Instr. in Civ. Engrg., Oklahoma State Univ., Stillwater, Okla.
[2] Prof. of Civ. Engrg., Oklahoma State Univ., Stillwater, Okla.

Kleinlogel,[3] K. Beyer,[4] R. Saliger,[5] Z. Bažant, et al,[6] A. Amerikian,[7] John D. Griffiths,[8] M. P. Korn,[9] and others.

For the analysis of continuous gable frames, algebraic formulas based on the principle of elastic deformation have been derived by Kleinlogel.[10]

Charts for the analysis of continuous gabel frames are presented herein. Two, three, and four equal-span frames of constant cross section subjected to uniformly distributed loads are considered. The bottoms of frames are assumed to be hinged and all dimensions are expressed in terms of the span and parameters α and β (Fig. 1).

All deformations are assumed to be elastic. The deformations due to normal forces and shearing forces are neglected. The sign convention of the slope-deflection method is introduced. The positive reactive moments and angular rotations are clockwise. The positive reactive forces are to the right and upward. The positive linear displacements are to the left.

The slope-deflection equations used in this study were formulated by A. Vasquez Pena[11] and F. Hedges.[12] The standard nomenclature is used throughout the paper. The meaning of special symbols is explained wherever they occur.

NOMENCLATURE

M_{ij} = Moment at i on member ij;

H_{ij} = Horizontal thrust at i on member ij;

V_{ij} = Shear at i on member ij;

α = Column parameter;

β = Gable height parameter;

γ = $\sqrt{1 + 4\beta^2}$;

θ_i = Angular rotation at i;

Δ_{ix} = Horizontal displacement at i;

Δ_{ijx} = Relative horizontal displacement $\Delta_{ix} - \Delta_{jx}$;

w = Intensity of load; and

Q_{ij} = Moment coefficient for M_{ij}.

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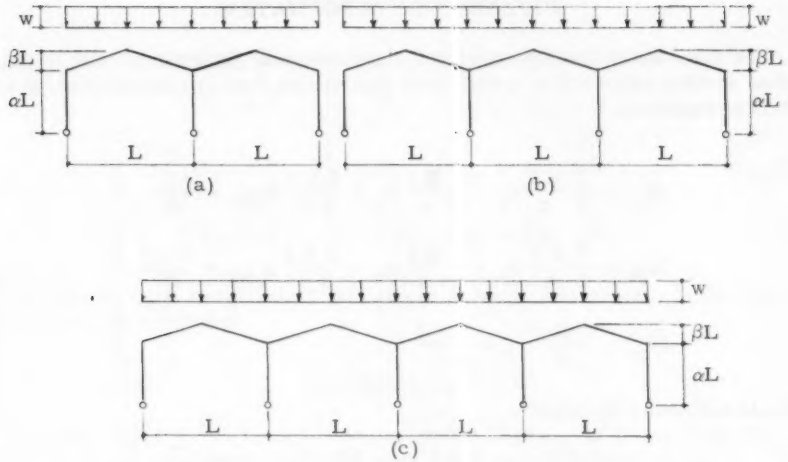

FIG. 1.—CONTINUOUS GABLE FRAMES

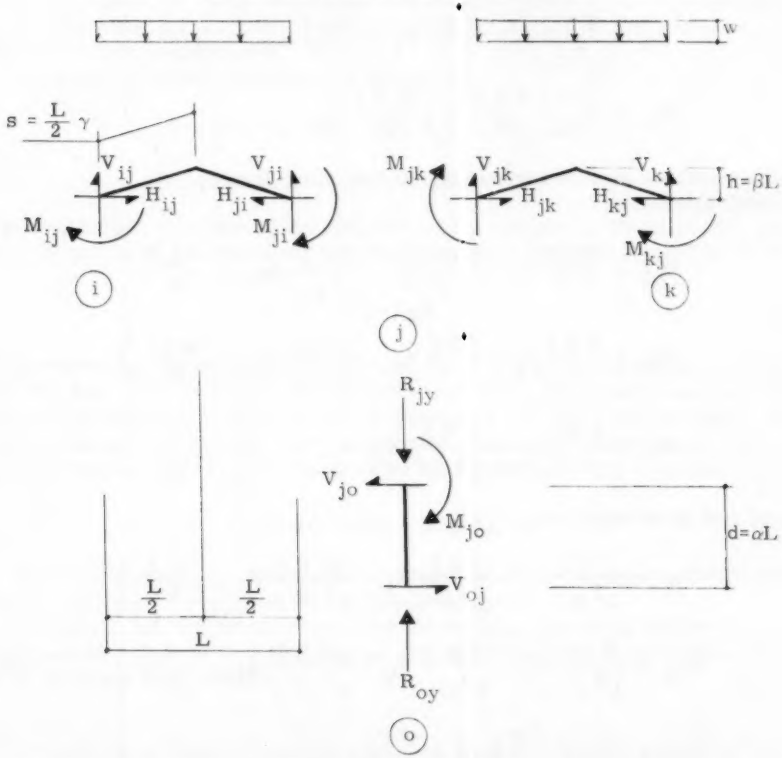

FIG. 2.—A TYPICAL JOINT

EQUATIONS OF DEFORMATION

The slope deflection equations for a symmetrical gable frame with constant cross section subjected to a uniformly distributed load are stated (Fig. 2) as:
Moment equations:

$$\left.\begin{aligned} M_{ji} &= \frac{7}{2}\frac{E\,I}{s}\theta_j - \frac{1}{2}\frac{E\,I}{s}\theta_i + \frac{3\,E\,I}{s\,h}\Delta_{ijx} + \frac{w\,L^2}{48} \\ M_{jk} &= \frac{7}{2}\frac{E\,I}{s}\theta_j - \frac{1}{2}\frac{E\,I}{s}\theta_k - \frac{3\,E\,I}{s\,h}\Delta_{jkx} - \frac{w\,L^2}{48} \\ M_{jo} &= \frac{3\,E\,I}{d}\theta_j + \frac{3\,E\,I}{d^2}\Delta_{jx} \end{aligned}\right\} \quad \ldots\ldots (1a)$$

Thrust and shear equations:

$$\left.\begin{aligned} H_{ji} &= \frac{3\,E\,I}{s\,h}\theta_i - \frac{3\,E\,I}{s\,h}\theta_j - \frac{6\,E\,I}{s\,h^2}\Delta_{ijx} + \frac{w\,L^2}{8\,h} \\ H_{jk} &= \frac{3\,E\,I}{s\,h}\theta_j - \frac{3\,E\,I}{s\,h}\theta_k - \frac{6\,E\,I}{s\,h^2}\Delta_{jkx} + \frac{w\,L^2}{8\,h} \\ V_{jo} &= V_{oj} = \frac{3\,E\,I}{d^2}\theta_j + \frac{3\,E\,I}{d^3}\Delta_{jx} \end{aligned}\right\} \quad \ldots\ldots (2a)$$

These equations may be expressed in terms of α, β, and γ.
Moment equations:

$$\left.\begin{aligned} M_{ji} &= \frac{7\,E\,I}{L\,\gamma}\theta_j - \frac{1\,E\,I}{L\,\gamma}\theta_i + \frac{6\,E\,I}{L^2\,\beta\,\gamma}\Delta_{ijx} + \frac{w\,L^2}{48} \\ M_{jk} &= \frac{7\,E\,I}{L\,\gamma}\theta_j - \frac{1\,E\,I}{L\,\gamma}\theta_k - \frac{6\,E\,I}{L^2\,\beta\,\gamma}\Delta_{jkx} - \frac{w\,L^2}{48} \\ M_{jo} &= \frac{3\,E\,I}{L\,\alpha}\theta_j + \frac{3\,E\,I}{L^2\,\alpha^2}\Delta_{jx} \end{aligned}\right\} \quad \ldots\ldots (1b)$$

Thrust and shear equations:

$$\left.\begin{aligned} H_{ji} &= \frac{6\,E\,I}{L^2\,\beta\,\gamma}\theta_i - \frac{6\,E\,I}{L^2\,\beta\,\gamma}\theta_j - \frac{12\,E\,I}{L^3\,\beta^2\,\gamma}\Delta_{ijx} + \frac{w\,L}{8\,\beta} \\ H_{jk} &= \frac{6\,E\,I}{L^2\,\beta\,\gamma}\theta_j - \frac{6\,E\,I}{L^2\,\beta\,\gamma}\theta_k - \frac{12\,E\,I}{L^3\,\beta^2\,\gamma}\Delta_{jkx} + \frac{w\,L}{8\,\beta} \\ V_{jo} &= V_{oj} = \frac{3\,E\,I}{L^2\,\alpha^2}\theta_j + \frac{3\,E\,I}{L^3\,\alpha^3}\Delta_{jx} \end{aligned}\right\} \quad \ldots\ldots (2b)$$

JOINT EQUATIONS

Three conditions of static equilibrium must be satisfied at each joint:

$$\sum M_j = 0 \quad | \quad M_{ji} + M_{jo} + M_{jk} = 0 \ldots\ldots\ldots \quad (3)$$

$$\sum F_{jx} = 0 \quad | \quad -H_{ji} - V_{jo} + H_{jk} = 0 \ldots\ldots\ldots \quad (4)$$

$$\sum F_{jy} = 0 \quad | \quad V_{ji} - R_{jy} + V_{jk} = 0 \ldots\ldots\ldots \quad (5)$$

In the case of rigid foundation at the bottom of each column, only joint Eqs. 3 and 4 may be considered.

MATRIX TABLE

The joint Eqs. 3 and 4 have been expressed in terms of Eqs. 1b and 2b for all joints of the two, three, and four span gable frames shown in Fig. 1. The results were recorded in three matrix tables each corresponding to one frame. In the formation of these matrices, the reoccurrence of certain groups of terms was noted. This phenomena is due to the nature of the frame and the slope-deflection equation.

The solution of these matrices in terms of

$$\alpha = 0.2, 0.4, 0.5, 0.6, 0.7, 0.8, 1.0$$

$$\beta = 0.1, 0.2, 0.3, 0.4, 0.5$$

is accomplished by means of the IBM 650 digital computer. The computer program is on file in the school of Civil Engineering, Oklahoma State University.

CHARTS

The deformation equivalents obtained from the IBM 650 output have been used for the calculation of moment coefficients Q's. The results are recorded graphically (Charts A-20, A-21, A-30 through A-33, and A-40 through A-44). The charts can be utilixed very efficiently if β has a value that is charted. Quite accurate results can be obtained by linear interpolation from the charts.

NUMERICAL EXAMPLES

Three numerical examples are introduced to illustrate the numerical procedure. All values are given in feet, kips, or kip-feet.

Example 1.—A two-equal-span gable frame (Fig. 3) is analyzed by means of Charts A-20 and A-21 and the results are compared with those obtained[13] by M. P. Korn and J. S. Kinney.

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13 "Steel Rigid Frames Manual-Analysis of Multispan Rigid Frames," by M. P. Korn and J. S. Kinney, Ann Arbor, Mich., 1953, pp. 59-65.

(a) Parameters:

$$\alpha = \frac{15}{40} = 0.375 \quad \left| \quad \beta = \frac{10}{40} = 0.250 \right. \quad \left| \quad w\,L^2 = 3{,}200 \right.$$

(b) End moments:

From Charts	From Korn and Kinney
$M_{12} = -(0.0580)(3{,}200) = -185.60$	$M_{12} = -187.80$
$M_{21} = +(0.0663)(3{,}200) = +212.16$	$M_{21} = +215.40$

Example 2.—A three-equal-span gable frame (Fig. 4) is analyzed by means of Charts A-30 through A-33 and the results are compared with those of F. Hedges.[12]

(a) Parameters:

$$\alpha = \frac{20}{40} = 0.500 \quad \left| \quad \beta = \frac{10}{40} = 0.250 \right. \quad \left| \quad w\,L^2 = 1{,}920 \right.$$

(b) End moments:

From Charts	From Hedges
$M_{12} = -(0.0605)(1{,}920) = -116.16$	$M_{12} = -116.60$
$M_{21} = +(0.0682)(1{,}920) = +130.94$	$M_{21} = +132.60$
$M_{20} = +(0.0020)(1{,}920) = +\ 3.84$	$M_{20} = +\ 3.00$
$M_{23} = -(0.0702)(1{,}920) = -134.78$	$M_{23} = -135.00$

Example 3.—A four-equal-span gable frame (Fig. 5) is analyzed by means of Charts A-40 through A-44, and the results are compared with those of J. J. Tuma et al.[14]

(a) Parameters:

$$\alpha = \frac{19}{40} = 0.475 \quad \left| \quad \beta = \frac{6}{40} = 0.150 \right. \quad \left| \quad w\,L^2 = 1{,}216 \right.$$

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14 "Analysis of Frames with Curved and Bent Members," by J. J. Tuma, K. S. Havner and F. Hedges, Proceedings, ASCE, Vol. 84, No. ST 5, September, 1958, pp. 19-22.

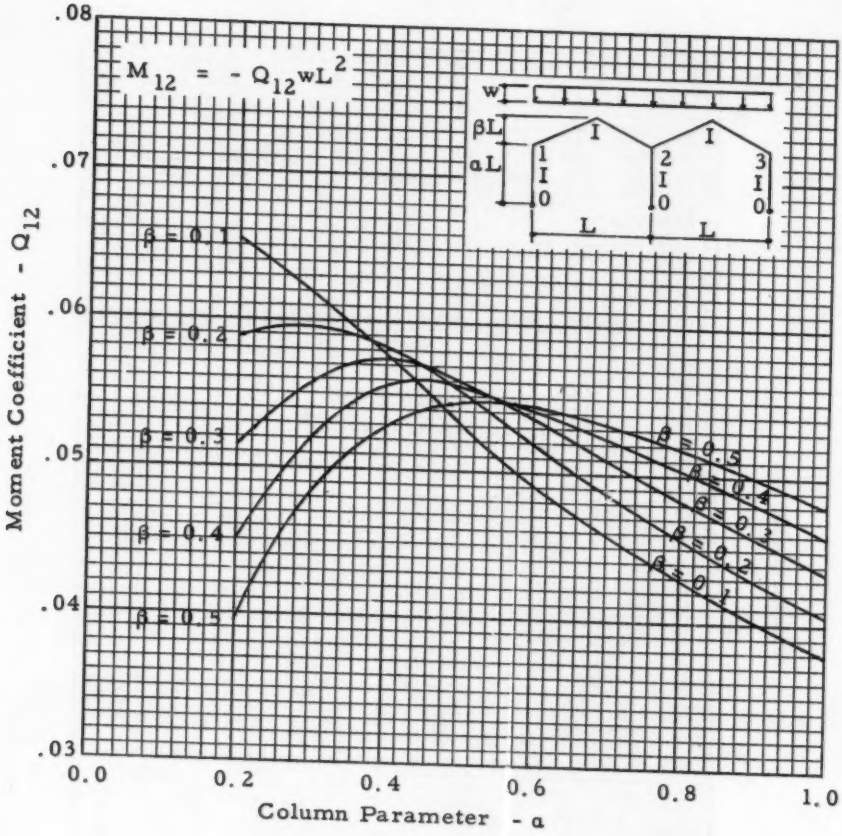

CHART A-20.—TWO-SPAN FRAME, BENT GIRDERS

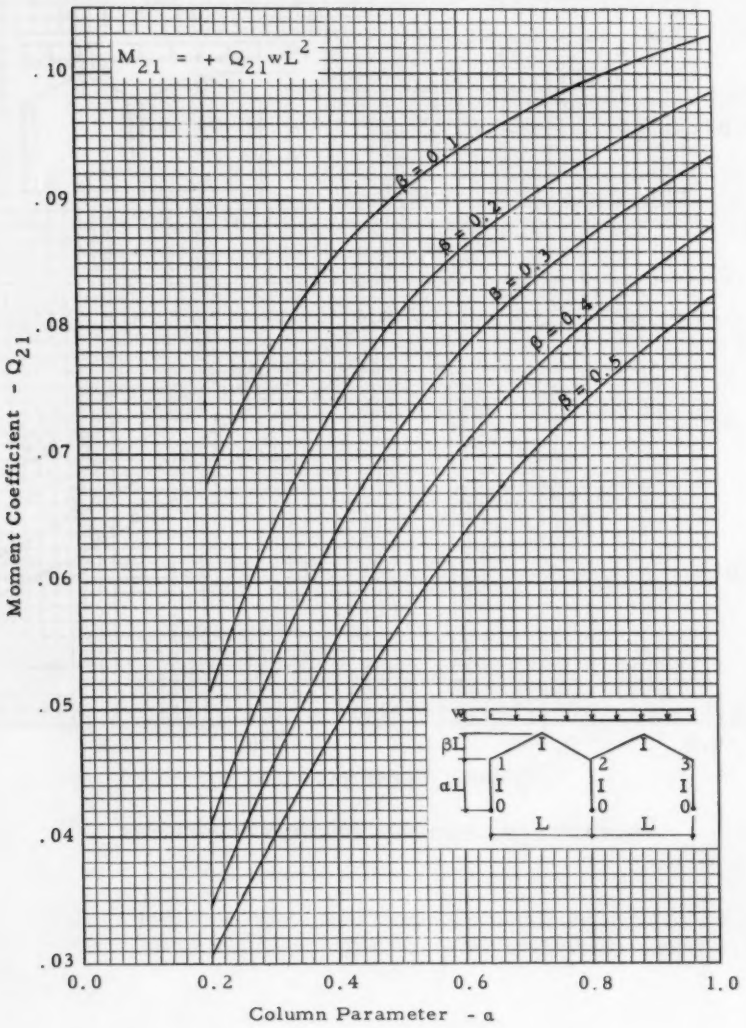

CHART A-21.—TWO SPAN FRAME, BENT GIRDERS

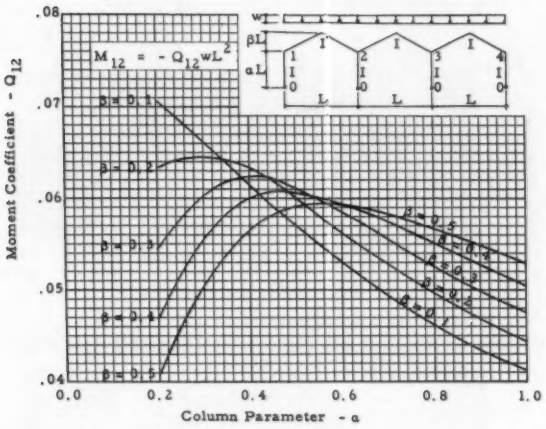

CHART A-30.—THREE-SPAN FRAME, BENT GIRDERS

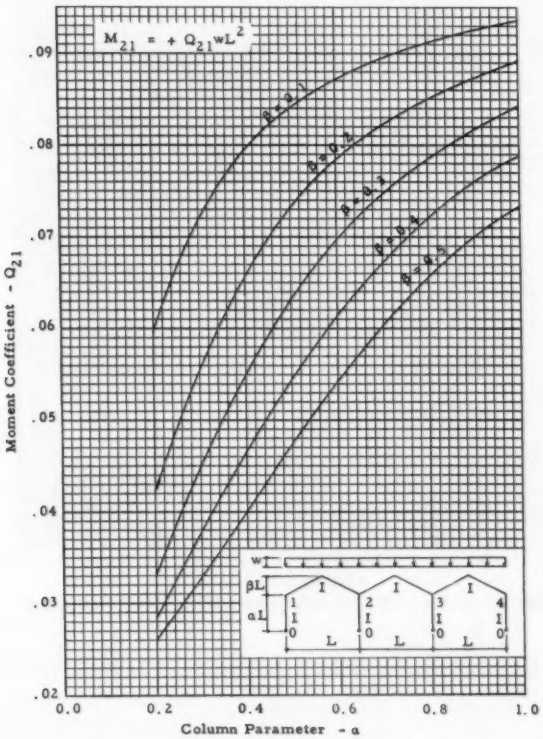

CHART A-31.—THREE-SPAN FRAME, BENT GIRDERS

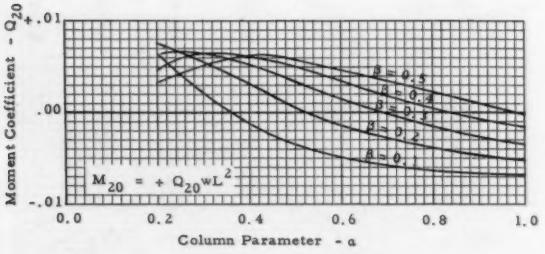

CHART A-32.- THREE-SPAN FRAME, BENT GIRDERS

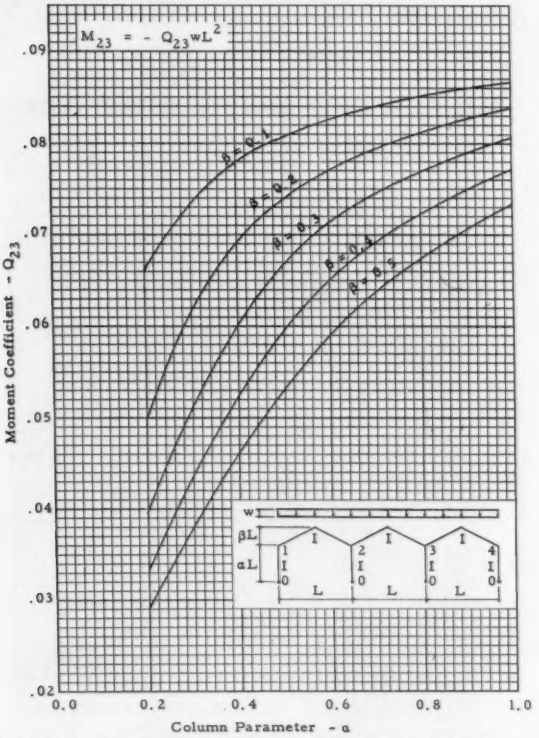

CHART A-33.—THREE-SPAN FRAME, BENT GIRDERS

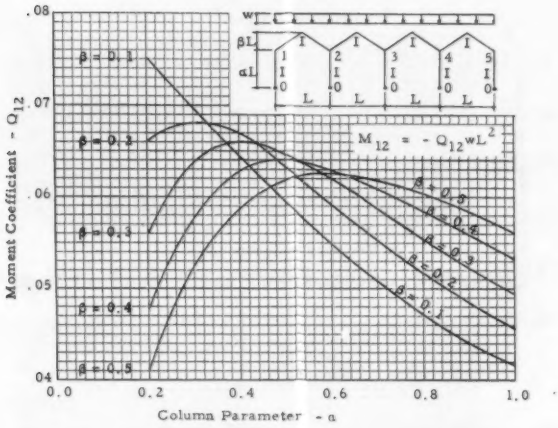

CHART A-40.—FOUR-SPAN FRAME, BENT GIRDERS

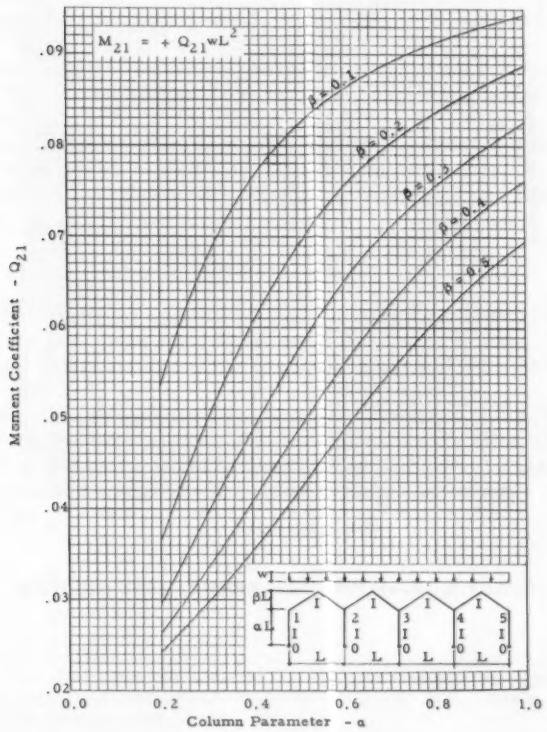

CHART A-41.—FOUR-SPAN FRAME, BENT GIRDERS

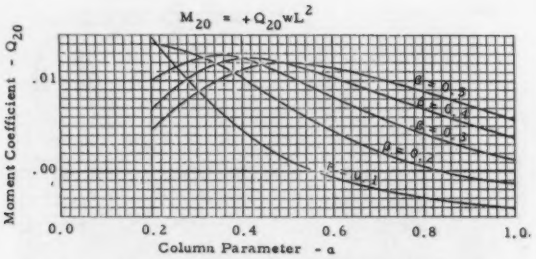

CHART A-42.—FOUR-SPAN FRAME, BENT GIRDERS

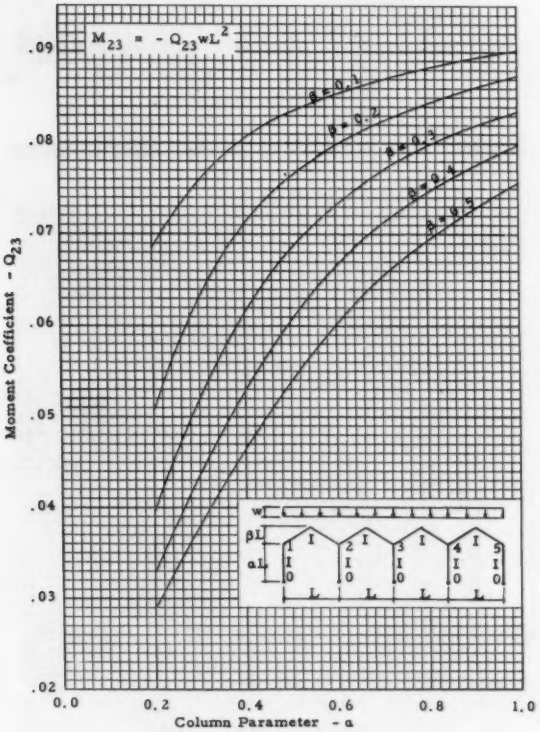

CHART A-43.—FOUR-SPAN FRAME, BENT GIRDERS

(b) End moments:

From Tables A-40 through A-44	From Tuma, et al
$M_{12} = -(0.0625)(1,216) = -76.00$	$M_{12} = -76.34$
$M_{21} = (0.0739)(1,216) = +89.86$	$M_{21} = +89.73$
$M_{20} = +(0.0050)(1,216) = + 6.08$	$M_{20} = + 6.55$
$M_{23} = -(0.0789)(1,216) = -95.94$	$M_{23} = -96.27$
$M_{32} = +(0.0692)(1,216) = +84.15$	$M_{32} = +84.25$

SUMMARY

(1) It has been the purpose of this paper to present charts which will give end moments for continuous gable frames of constant cross section loaded by a uniformly distributed load.

(2) The limits of the parameters α and β were chosen to include all combinations of span length, column length, and gable height that are thought to be practical.

(3) The use of these charts is illustrated (Examples 1,2,3), and the results indicate that they can be utilized very effectively for the preliminary analysis of these frames.

(4) Similar charts for other load conditions, variations of spans, and cross sections are being prepared and will be published elsewhere.[16]

(5) For the final analysis the procedure developed by Tuma, et al[12] is recommended.

ACKNOWLEDGMENTS

This paper is based on a thesis by J. W. Gillespie[15] submitted to the graduate school of the Oklahoma State University, Stillwater, Okla., in partial fulfillment of the requirements for the degree of Master of Science in Civil Engineering.

The investigation was performed under the direction of J. J. Tuma, and was supported by the Robberson Steel Company of Oklahoma City, Okla. The reported moment coefficients were compiled on the IBM 650 electronic digital computer at the Oklahoma State University computing center.

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15 "Tables and Nomographic Charts for the Preliminary Analysis of Continuous Rigid Frames with Bent Members," by J. W. Gillespie, M.S. Thesis, Oklahoma State Univ. Library, Stillwater, Okla., 1958.

16 "Charts for Preliminary Analysis of Continuous Gable Frames," by J. W. Gillespie, Oklahoma State University Engineering Experiment Station Publication, Stillwater (in preparation).

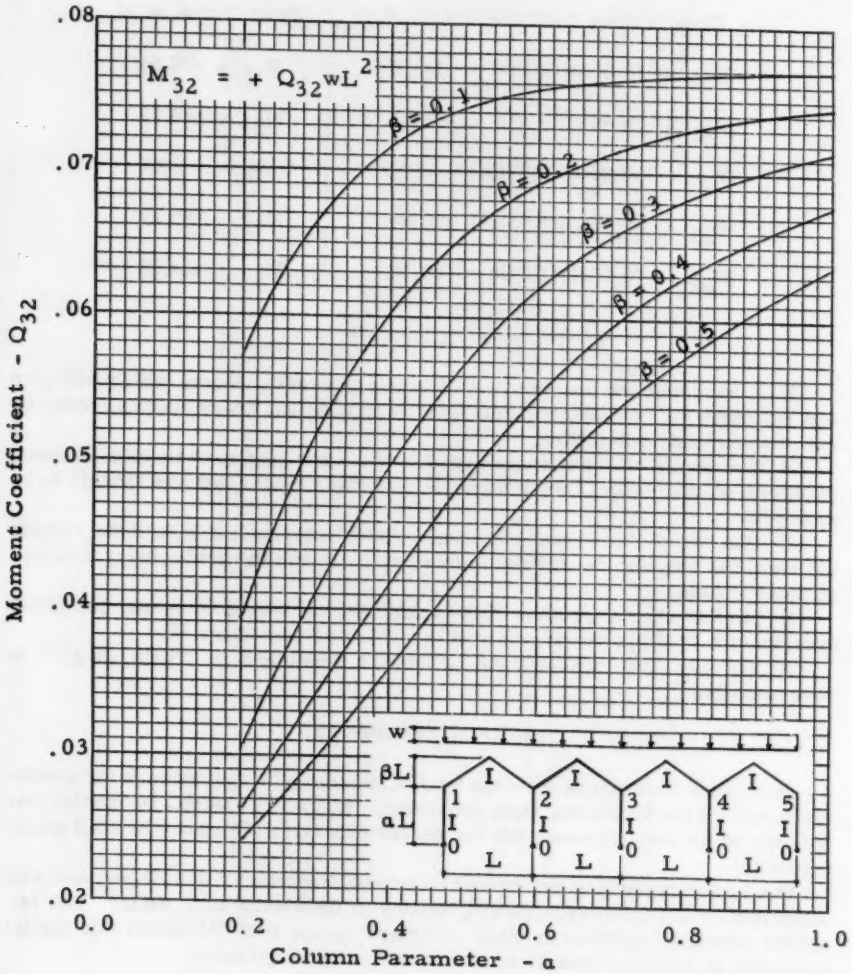

CHART A 44.—FOUR-SPAN FRAME, BENT GIRDERS

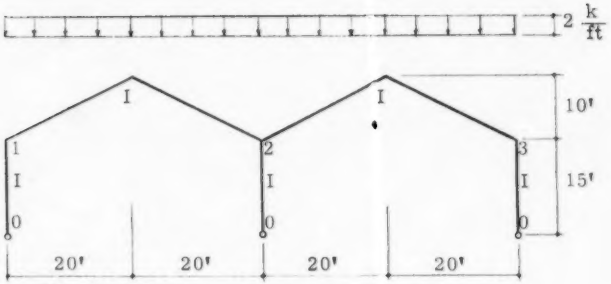

FIG. 3.—TWO-SPAN GABLE FRAME

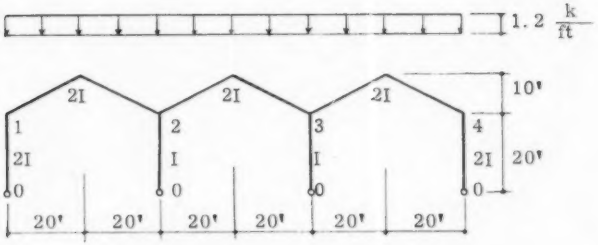

FIG. 4.—THREE-SPAN GABLE FRAME

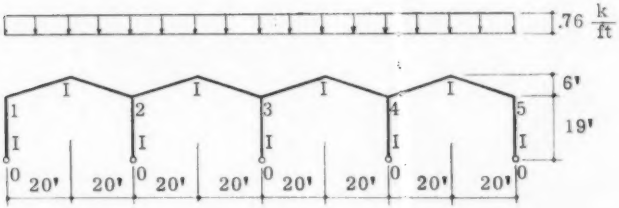

FIG. 5.—FOUR-SPAN GABLE FRAME

PROCEEDINGS PAPERS

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VOLUME 85 (1959)

APRIL: 1990(EM2), 1991(EM2), 1992(EM2), 1993(HW2), 1994(HY4), 1995(HY4), 1996(HY4), 1997(HY4), 1998 (SM2), 1999(SM2), 2000(SM2), 2001(SM2), 2002(ST4), 2003(ST4), 2004(ST4), 2005(ST4), 2006(PO2), 2007 (HW2)[c], 2008(EM2)[c], 2009(ST4)[c], 2010(SM2)[c], 2011(SM2)[c], 2012(HY4)[c], 2013(PO2)[c].

MAY: 2014(AT2), 2015(AT2), 2016(AT2), 2017(HY5), 2018(HY5), 2019(HY5), 2020(HY5), 2021(HY5), 2022(HY5), 2023(PL2), 2024(PL2), 2025(PL2), 2026(PP1), 2027(PP1), 2028(PP1), 2029(PP1), 2030(SA3), 2031(SA3), 2032(SA3), 2033(SA3), 2034(ST5), 2035(ST5), 2036(ST5), 2037(ST5), 2038(PL2), 2039(PL2), 2040(AT2)[c]. 2041(PL2)[c], 2042(PP1)[c], 2043(ST5)[c], 2044(SA3)[c], 2045(HY5)[c], 2046(PP1), 2047(PP1).

JUNE: 2048(CP1), 2049(CP1), 2050(CP1), 2051(CP1), 2052(CP1), 2053(CP1), 2054(CP1), 2055(CP1), 2056 (HY6), 2057(HY6), 2058(HY6), 2059(IR2), 2060(IR2), 2061(PO3), 2062(SM3), 2063(SM3), 2064(SM3), 2065 (ST6), 2066(WW2), 2067(WW2), 2068(WW2), 2069(WW2), 2070(WW2), 2071(WW2), 2072(CP1)[c], 2073(IR2)[c], 2074(PO3)[c], 2075(ST6)[c], 2076(HY6)[c], 2077(SM3)[c], 2078(WW2)[c].

JULY: 2079(HY7), 2080(HY7), 2081(HY7), 2082(HY7), 2083(HY7), 2084(HY7), 2085(HY7), 2086(SA4), 2087 (SA4), 2088(SA4), 2089(SA4), 2090(SA4), 2091(EM3), 2092(EM3), 2093(EM3), 2094(EM3), 2095(EM3), 2096 (EM3), 2097(HY7)[c], 2098(SA4)[c], 2099(EM3)[c], 2100(AT3), 2101(AT3), 2102(AT3), 2103(AT3), 2104(AT3), 2105(AT3), 2106(AT3), 2107(AT3), 2108(AT3), 2109(AT3), 2110(AT3), 2111(AT3), 2112(AT3), 2113(AT3), 2114(AT3), 2115(AT3), 2116(AT3), 2117(AT3), 2118(AT3), 2119(AT3), 2120(AT3), 2121(AT3), 2122(AT3), 2123(AT3), 2124(AT3), 2125(AT3).

AUGUST: 2126(HY8), 2127(HY8), 2128(HY8), 2129(HY8), 2130(PO4), 2131(PO4), 2132(PO4), 2133(PO4), 2134 (SM4), 2135(SM4), 2136(SM4), 2137(SM4), 2138(HY8)[c], 2139(PO4)[c] 2140(SM4)[c].

SEPTEMBER: 2141(CO2), 2142(CO2), 2143(CO2), 2144(HW3), 2145(HW3), 2146(HW3), 2147(HY9), 2148(HY9), 2149(HY9), 2150(HY9), 2151(IR3), 2152(ST7)[c], 2153(IR3), 2154(IR3), 2155(IR3), 2156(IR3), 2157(IR3), 2158 (IR3), 2159(IR3), 2160(IR3), 2161(SA5), 2162(SA5), 2163(ST7), 2164(ST7), 2165(SU1), 2166(SU1), 2167(WW3), 2168(WW3), 2169(WW3), 2170(WW3), 2171(WW3), 2172(WW3), 2173(WW3), 2174(WW3), 2175(WW3), 2176 (WW3), 2177(WW3), 2178(CO2)[c], 2179(IR3)[c], 2180(HW3)[c], 2181(SA5)[c], 2182(HY9)[c], 2183(SU1)[c], 2184 (WW3)[c], 2185(PP2)[c], 2186(ST7)[c], 2187(PP2), 2188(PP2).

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DECEMBER: 2271(HY12)[c], 2272(CP2), 2273(HW4), 2274(HW4), 2275(HW4), 2276(HW4), 2277(HW4), 2278 (HW4), 2279(HW4), 2280(HW4), 2281(IR4), 2282(IR4), 2283(IR4), 2284(IR4), 2285(PO6), 2286(PO6), 2287 (PO6), 2288(PO6), 2289(PO6), 2290(PO6), 2291(PO6), 2292(SM6), 2293(SM6), 2294(SM6), 2295(SM6), 2296 (SM6), 2297(WW4), 2298(WW4), 2299(WW4), 2300(WW4), 2301(WW4), 2302(WW4), 2303(WW4), 2304(HW4), 2305(ST10), 2306(CP2), 2307(CP2), 2308(ST10), 2309(CP2), 2310(HY12), 2311(HY12), 2312(PO6), 2313(PO6), 2314(ST10), 2315(HY12), 2316(HY12), 2317(HY12), 2318(WW4), 2319(SM6), 2320(SM6), 2321(ST10), 2322 (ST10), 2323(HW4)[c]. 2324(CP2)[c], 2325(SM6)[c], 2326(WW4)[c], 2327(IR4)[c], 2328(PO6)[c], 2329(ST10)[c], 2330 (CP2).

VOLUME 86 (1960)

JANUARY: 2331(EM1), 2332(EM1), 2333(EM1), 2334(EM1), 2335(HY1), 2336(HY1), 2337(EM1), 2338(EM1), 2339(HY1), 2340(HY1), 2341(SA1), 2342(EM1), 2343(SA1), 2344(ST1), 2345(ST1), 2346(ST1), 2347(ST1), 2348(EM1)[c], 2349(HY1)[c], 2350(ST1), 2351(ST1), 2352(SA1)[c], 2353(ST1)[c], 2354(ST1).

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MARCH: 2393(IR1), 2394(IR1), 2395(IR1), 2396(IR1), 2397(IR1), 2398(IR1), 2399(IR1), 2400(IR1), 2401(IR1), 2402(IR1), 2403(IR1), 2404(IR1), 2405(IR1), 2406(IR1), 2407(SA2), 2408(SA2), 2409(HY3), 2410(ST3), 2411 (SA2), 2412(HW1), 2413(WW1), 2414(WW1), 2415(HY3), 2416(HW1), 2417(HW3), 2418(HW1)[c], 2419(WW1)[c], 2420(WW1), 2421(WW1), 2422(WW1), 2423(WW1), 2424(SA2), 2425(SA2)[c], 2426(HY3)[c], 2427(ST3)[c]

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c. Discussion of several papers, grouped by divisions.

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